Electromagnetics with finite elements: static problems

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PREFACE

The book describes application of the finite element method to electro- and magnetostatic problems. The main emphasis of the book is on the electromagnetic formulations rather than on the diverse features of the finite element method. Boundary value problems based on the div-grad and curl-curl equations, are discussed in great detail. All derivation steps on the way from Maxwell's equations to the numerical recipes that can be programmed into a computer are discussed at length. A special attention is paid to boundary and interface conditions.

The book is supplemented with computer code written with a help of deal.II finite element library. The computer code illustrates application of the finite element method to various electromagnetic formulations discussed in the book. The documentation of the computer code exists in the form of an online book at www.cembooks.nl. The computer code can be downloaded at www.cembooks.nl or, alternatively, at www.github.com/cembooks.

The finite element method is described as seen by a user of the deal.II finite element library. The mesh cells are assumed to be quadrilateral and hexahedral in two- and three-dimensional spaces, respectively. The four finite elements essential for the De Rham complex are assumed to be the finite elements available in deal.II, i.e., Lagrange, Nedelec, Raviart-Thomas, and discontinuous Lagrange finite elements. All dielectric and magnetic materials are assumed to be linear, lossless, and isotropic.

The book is organized as the following.

Chapter 1 contains a description of the basic concepts that are essential for formulating problems in electromagnetics and for understanding the finite element method.

Chapter 2 is dedicated to the formulation of the static scalar boundary value problem that can describe most of the problems in the electro- and magnetostatics. Next, a number of useful techniques that can help to make a successful computer code are considered.

Chapter 3 deals with formulations of magnetostatic problems. Boundary value problems formulated in therms of the total scalar magnetic potential, reduced scalar magnetic potential, and vector magnetic potential are derived. Some useful techniques for reducing the size of the problem domain as well as for dealing with the unbounded problem domain are discussed as well.

Chapter 4 deals with variational formulations. The static scalar boundary value problem derived in Chapter 2 and the static vector boundary value problem derived in Chapter 3 are converted into functionals. In the end of this chapter a general functional for projections is derived. A projection is, essentially, an operation that helps to place a physical quantity into a correct function space.

Chapter 5 introduces the finite element method. Four types of finite elements are discussed. The Bossavit's diagrams are presented as a great tool for describing problems in electromagnetics. Two methods of allocating a correct type of finite elements to a physical quantity are described. The functionals that are derived in Chapter 4 are converted into numerical recipes. A number of useful projection operations are converted into numerical recipes as well. The numerical recipes are, essentially, the formulas that are implemented by the solvers and projectors of the supplemented computer code.

Chapter 6 is a collection of solved problems in electromagnetics that have closed-form analytical solutions. The first thought that naturally appears after programming a numerical recipe into a computer is: does it really work? One way to answer this question is to feed the code a problem and compare the numerical result to a known closed-form analytical solution. One problem, however, cannot cover all aspects of a boundary value problem. It is better to have a collection of problems. Chapter 6 contains such collection.

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Rotterdam, 2024

Chapter 1

Preliminaries

1.1 Vector calculus

1.1.1 Vectors

The vectors as they are known today in electromagnetics were introduced in the end of the nineteenth century by Oliver Heaviside. In [23], page 132, he wrote,

"Ordinary algebra, as is well known, treats of quantities and their relations. If, however, we examine geometry, we shall soon find that the fundamental quantity concerned, namely a straight line, when regarded as an entity, cannot be treated simply as a quantity in the algebraical sense. It has, indeed, size, viz., its length; but with this is conjoined another important property, its direction. Taken as a whole, it is a Vector. In contrast with this, an ordinary quantity, having size only, is a Scalar."

Following the footsteps of Oliver Heaviside, we define a vector as a magnitude and direction grouped together.

There are two ways to approach problems involving vectors. The first way is the way of geometry. The second way is the way of algebra. Let us consider the problem of adding two vectors in a two-dimensional space as an example:

$$\vec{C} = \vec{A} + \vec{B}. \tag{1.1.1}$$

The geometric approach to the problem suggests that we need to draw three vectors as shown in Figure 1.1.1 A) and reason that the identity given by equation (1.1.1) holds as the result of following vectors $\vec{A}$ and $\vec{B}$ in Figure 1.1.1 A) is the same as the result of following vector $\vec{C}$: both paths lead us from point P_1 to point P_2. The three vectors, i.e., $\vec{A}$, $\vec{B}$, and $\vec{C}$, form a triangle. Then the length of the side of the triangle formed by vector $\vec{C}$ can be deduced from the law of cosines as

$$C = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos(\alpha)}. \tag{1.1.2}$$

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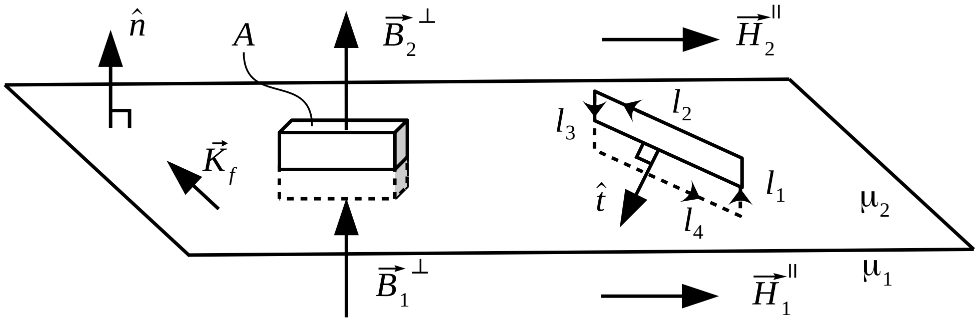

Figure 1.3.8: Interface between two dissimilar soft magnetic materials.

Second, we apply the main theorems for divergence (1.1.46) and curl (1.1.45) to the first and the second equation, respectively:

$$\begin{aligned} &\oiint_S \vec{B}\cdot d\vec{S} = 0 \quad \text{(i)}, \\ &\oint_L \vec{H}\cdot d\vec{l} = I_f \quad \text{(ii)}. \end{aligned} \tag{1.3.24}$$

Here S is the closed surface that bounds the volume V, L is the closed loop that bounds the surface S', and I_f is the total free current that crosses the open surface S'.

Next, we will derive the interface conditions for the magnetostatic field by considering the interface between two dissimilar soft magnetic materials. The interface is depicted in Figure 1.3.8. Let us consider the Gaussian box on the left side of the figure. The box is assumed to be so small that the magnetic field above the interface, $\vec{B}_2$, is constant all over the box. The same can be said about the magnetic field below the interface, $\vec{B}_1$. Therefore, the net flux of the magnetic field through the vertical facets of the box equals zero. Then to satisfy the identity (i) in (1.3.24), the fluxes of the magnetic field through the horizontal facets of the box must balance each other out:

$$A\hat{n}\cdot\vec{B}_1 = A\hat{n}\cdot\vec{B}_2.$$

The scalar A in the last equation is the area of the top or the bottom facet of the box. The last equation can be simplified as

$$\hat{n}\cdot\vec{B}_1 = \hat{n}\cdot\vec{B}_2. \tag{1.3.25}$$

Next, let us consider the Amperian loop depicted in the right half of Figure 1.3.8. The loop is so small that the field $\vec{H}_2$ above the interface is constant across the loop. The same can be said about the vector field $\vec{H}_1$ below the interface. We can split the integral in the second equation of (1.3.24) in four parts like so:

$$\oint \vec{H}\cdot d\vec{l} = \int \vec{H}\cdot d\vec{l}_1 + \int \vec{H}\cdot d\vec{l}_2 + \int \vec{H}\cdot d\vec{l}_3 + \int \vec{H}\cdot d\vec{l}_4 = I_f. \tag{1.3.26}$$

As soon as $\vec{H}_1$ and $\vec{H}_2$ are assumed to be constant across the loop, the first and the third integrals cancel each other out. Then equation (1.3.26) can be rewritten as

$$\oint \vec{H}\cdot d\vec{l} = \underbrace{\int \vec{H}\cdot d\vec{l}_2}_{I_2} + \underbrace{\int \vec{H}\cdot d\vec{l}_4}_{I_4} = I_f.$$

Then assuming that the segments l_2 and l_4 are L meters long, we rewrite the last equation as

$$\underbrace{-\vec{H}_2\cdot(\hat{n}\times\hat{t})L}_{I_2} + \underbrace{\vec{H}_1\cdot(\hat{n}\times\hat{t})L}_{I_4} = \underbrace{\vec{K}_f\cdot\hat{t}L}_{I_f}. \tag{1.3.27}$$

The unit vector $\hat{t}$ in the last equation is normal to the loop such that

$$-\frac{d\vec{l}_2}{\left|d\vec{l}_2\right|} = \frac{d\vec{l}_4}{\left|d\vec{l}_4\right|} = \hat{n}\times\hat{t},$$

see Figure 1.3.8. By observing (1.1.7) we simplify equation (1.3.27) as the following

$$\Big(\big(\vec{H}_1-\vec{H}_2\big)\times\hat{n}\Big)\cdot\hat{t} = \vec{K}_f\cdot\hat{t}. \tag{1.3.28}$$

In other circumstances just discarding $\hat{t}$ in the last equation can be risky: the fact that projections of the two vectors, $\big(\vec{H}_1-\vec{H}_2\big)\times\hat{n}$ and $\vec{K}_f$, on the unit vector $\hat{t}$ are equal does not imply that the vectors themselves are equal. Equation (1.3.28), however, must hold for any choice of $\hat{t}$. Thai is, if we rotate the loop such that $\hat{t}$ draws a circle on the plane tangential to the interface, equation (1.3.28) must hold for all rotated instances of the loop. The last is only possible if

$$\big(\vec{H}_1-\vec{H}_2\big)\times\hat{n} = \vec{K}_f,$$

or, what is the same, see (i) in (1.1.5),

$$\hat{n}\times\vec{H}_2 - \hat{n}\times\vec{H}_1 = \vec{K}_f. \tag{1.3.29}$$

Then we can combine the last equation with (1.3.25) to write down the final version of the conditions on an interface between two dissimilar magnetic materials

$$\begin{array}{ll} \hat{n}\cdot\vec{B}_2 = \hat{n}\cdot\vec{B}_1 & \text{(i)}, \\ \hat{n}\times\vec{H}_2 - \hat{n}\times\vec{H}_1 = \vec{K}_f & \text{(ii)}. \end{array} \tag{1.3.30}$$

1.4 Linear algebra

In this section we extend the notion of vectors to column vectors. We denote a column vector by a lowercase boldface Latin letter:

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}.$$

We denote a matrix by an uppercase boldface letter,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

so a system of linear equations in the matrix form has the following appearance:

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{b}. \tag{1.4.1}$$

1.4.1 Two ways to perceive a matrix

The first way to perceive a matrix

The first way to perceive an $m \times n$ matrix is to see it as an ordered collection of n column vectors of a length m. There are four fundamental spaces associated with this perception of a matrix: column space $C(\boldsymbol{A})$, row space $C(\boldsymbol{A}^T)$, nullspace $N(\boldsymbol{A})$, and the left nullspace $N(\boldsymbol{A}^T)$. These four fundamental spaces constitute the big picture of linear algebra. It is shown in Figure 1.4.1. I have adapted it from Figure 3.5 that can be found on page 184 in [41]. Let us discuss it in more detail.

The column vector $\boldsymbol{b}$ in equation (1.4.1) can be interpreted as a linear combination of columns of the matrix $\boldsymbol{A}$:

$$\boldsymbol{b} = c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}. \tag{1.4.2}$$

Consequently, the column vector $\boldsymbol{b}$ can be perceived as an element of a vector space spanned by the columns of the matrix $\boldsymbol{A}$. This vector space is, essentially, the column space $C(\boldsymbol{A})$ in Figure 1.4.1. All column vectors in $C(\boldsymbol{A})$, including $\boldsymbol{b}$, have m components. The matrix $\boldsymbol{A}$ can be constructed such that absolutely any column vector that has m components can be represented as a linear combination of the columns of $\boldsymbol{A}$. In this case it is said that the columns of $\boldsymbol{A}$ span the entire vector space $\mathbb{R}^m$. That is, vector space $C(\boldsymbol{A})$ is a copy of $\mathbb{R}^m$. On the other hand, the matrix $\boldsymbol{A}$ can be constructed such that some of the vectors with m components cannot be represented as in (1.4.2). In this case, it is said that the column space $C(\boldsymbol{A})$ is a subspace of $\mathbb{R}^m$. The column vectors orthogonal to $C(\boldsymbol{A})$ are located in the left nullspace $N(\boldsymbol{A}^T)$. The left nullspace $N(\boldsymbol{A}^T)$ is an orthogonal complement of the column space $C(\boldsymbol{A})$. The last means that a set of m linearly independent column vectors chosen from $C(\boldsymbol{A})$ and $N(\boldsymbol{A}^T)$ span the entire vector space $\mathbb{R}^m$. Moreover, any vector in $C(\boldsymbol{A})$ is orthogonal to all vectors in $N(\boldsymbol{A}^T)$. In Figure 1.4.1 this

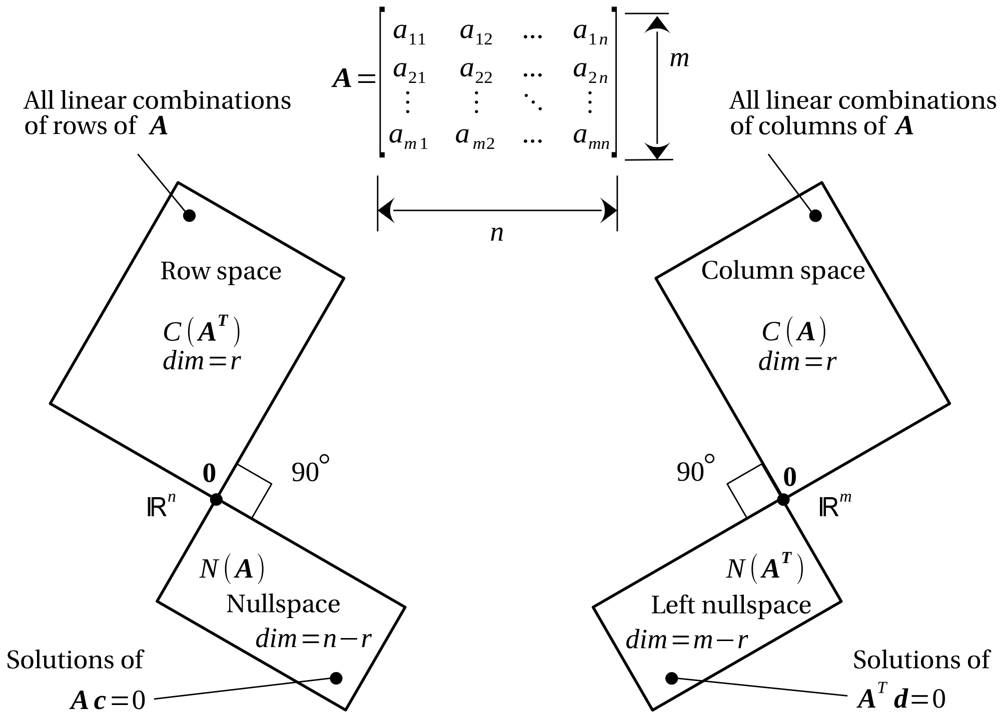

Figure 1.4.1: The four fundamental spaces. Adapted from [41]. The rectangles are schematic representations of multidimensional vector spaces. The 90° signs indicate that any pair of vectors taken from the respective spaces are orthogonal.

orthogonality is indicated by 90° sign. The column vectors that fill $N(\boldsymbol{A}^T)$ are, essentially, the solutions, $\boldsymbol{d}$, of the following system of linear equations:

$$\boldsymbol{A}^T \boldsymbol{d} = \boldsymbol{0}.$$

The left nullspace is never empty. There is always at least one vector in there, the **0** vector.

Likewise, the rows of the matrix $\boldsymbol{A}$ span the row space $C(\boldsymbol{A}^T)$. The solutions, $\boldsymbol{c}$, of the following system of linear equations:

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{0}$$

span the nullspace $N(\boldsymbol{A})$. The two spaces, $C(\boldsymbol{A}^T)$ and $N(\boldsymbol{A})$, are orthogonal complements of each other. A set of n linearly independent column vectors chosen from $C(\boldsymbol{A}^T)$ and $N(\boldsymbol{A})$ span the entire vector space $\mathbb{R}^n$. Moreover, any vector in $C(\boldsymbol{A}^T)$ is orthogonal to all vectors in $N(\boldsymbol{A})$.

The maximal amount of linearly independent vectors that can be chosen in a vector space defines the dimension of the vector space. The column and row spaces, i.e., $C(\boldsymbol{A})$ and $C(\boldsymbol{A}^T)$, have the same dimension. It equals the rank, r, of the matrix $\boldsymbol{A}$. Consequently, the null space $N(\boldsymbol{A})$ and the left null space $N(\boldsymbol{A}^T)$ have dimensions $n - r$ and $m - r$, respectively.

We are primarily interested in symmetric matrices, i.e., the square matrices

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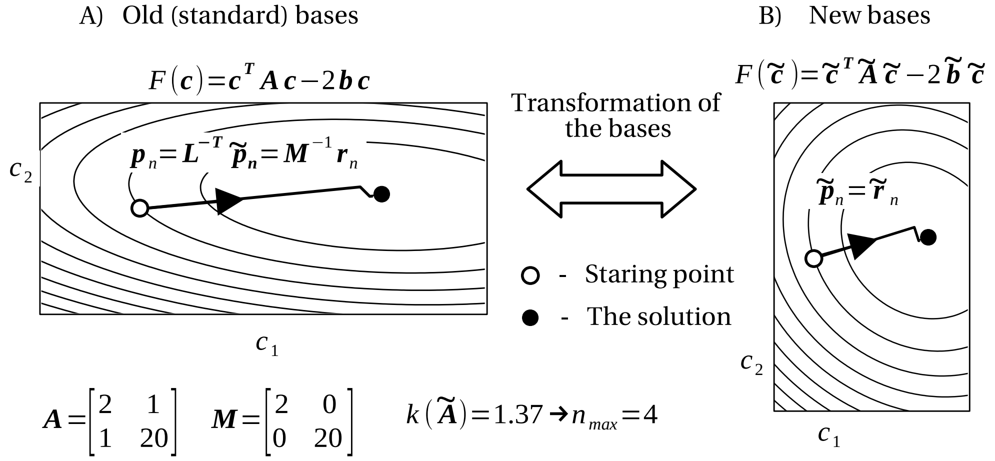

Figure 1.4.10: The same simulation of the steepest descent algorithm as shown in Figure 1.4.9 B) but with a preconditioner $\boldsymbol{M}$. The algorithm iterates in the old (standard) bases. The search direction is chosen in the new bases and transformed into the old (standard) bases.

have to be symmetric. Table 2.1. in [11] provides an overview of most widely used preconditioners.

Figure 1.4.10 illustrates an application of the steepest descent method with a preconditioner. The main diagonal of the system matrix in the standard basis, $\boldsymbol{A}$, has been chosen as a preconditioner, i.e, Jacobi. The example illustrates that the condition number of the system matrix as well as the total amount of iterations has decreased ten times due to preconditioning. Note that the shape of the contour curves of the quadratic form in the new bases, Figure 1.4.10 B), are closer to the shape of circles than the contour lines of the quadratic form in the old (standard) basis, Figure 1.4.10 A). That is the whole point of the preconditioning: to make the shape of the isoparametric curves of the quadratic form to be as close to the shape of circles as possible by changing the bases.

1.4.8 Conjugate gradient algorithm

The preconditioning considered in the preceding section improves the convergence rate of the steepest descent algorithm. This improvement can be explained as the following. In general, an iterative algorithm converges faster if the ellipsoids formed by the isoparametric surfaces of the quadratic form are of low eccentricity, see discussion on page 107. The preconditioning, on the other hand, can be interpreted as a change into dissimilar bases ($\tilde{\boldsymbol{E}} \neq \tilde{\boldsymbol{G}}$). The result of this bases change is the lower eccentricity of the ellipsoids formed by the isoparametric surfaces. For this reason, the steepest descent algorithm converges faster if iterated in the new (dissimilar) bases. In reality, however, the algorithm iterates in the old (standard) bases, but it does it so that the iteration steps being transformed into the new bases are the steps made by

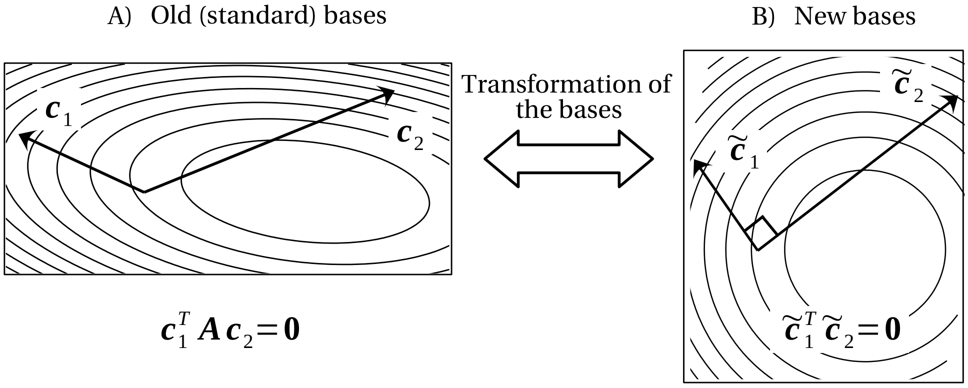

Figure 1.4.11: Two $\boldsymbol{A}$-conjugate vectors. It is possible to choose two vectors in the standard basis such that they are orthogonal to each other in the bases in which the isoparametric surfaces of the quadratic form are spheres.

the steepest descent algorithm without a preconditioner.

Ultimately, the convergence will be the fastest if the isoparametric surfaces are multidimensional spheres. In this case, the steepest descent algorithm will converge in just one step as the negative gradient of the quadratic form will point to the solution (the center of the spheres) at any starting point. A good question is: can we push the preconditioning trick to its limits? That is, while iterating in the standard basis, can we choose the search directions such as we were iterating in the bases in which the isoparametric surfaces of the quadratic form are spheres? Strictly speaking: no, we cannot. Such ultimate preconditioner would require $\boldsymbol{M}^{-1} = \boldsymbol{A}^{-1}$. On the other hand, if $\boldsymbol{A}^{-1}$ is known, there is no need to iterate: just compute the solution as $\boldsymbol{A}^{-1}\boldsymbol{b}$. However, the idea of iterating with the ultimate preconditioner is so attractive that we simply cannot waste it. We will attempt to apply it. This attempt will yield an algorithm known as conjugate gradient algorithm. In this section we will label the bases in which the isoparametric surfaces of the quadratic form are spheres as "new bases". By "old bases" we will mean the standard bases in which the system of linear equations is formulated. The main idea remains the same: we will iterate in the old bases, but choose the search directions wisely in the new bases. The conjugate gradient algorithm rests on two pillars: (i) the notion of $\boldsymbol{A}$-conjugate vectors and (ii) the search procedure in the new bases.

Let us consider the notion of $\boldsymbol{A}$-conjugate vectors first. We can choose two vectors in the old bases such that they are orthogonal in the new bases. This simple idea is illustrated in Figure 1.4.11. Indeed, the ideal preconditioner that makes an isoparametric surface of the quadratic form a sphere is $\boldsymbol{M} = \boldsymbol{A}$. That is, in this section we assume that

$$\boldsymbol{M} = \boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T, \tag{1.4.108}$$

see also equation (1.4.107). As it was the case with the preconditioner described in the preceding section, we choose the columns of $\boldsymbol{L}$ to be the new

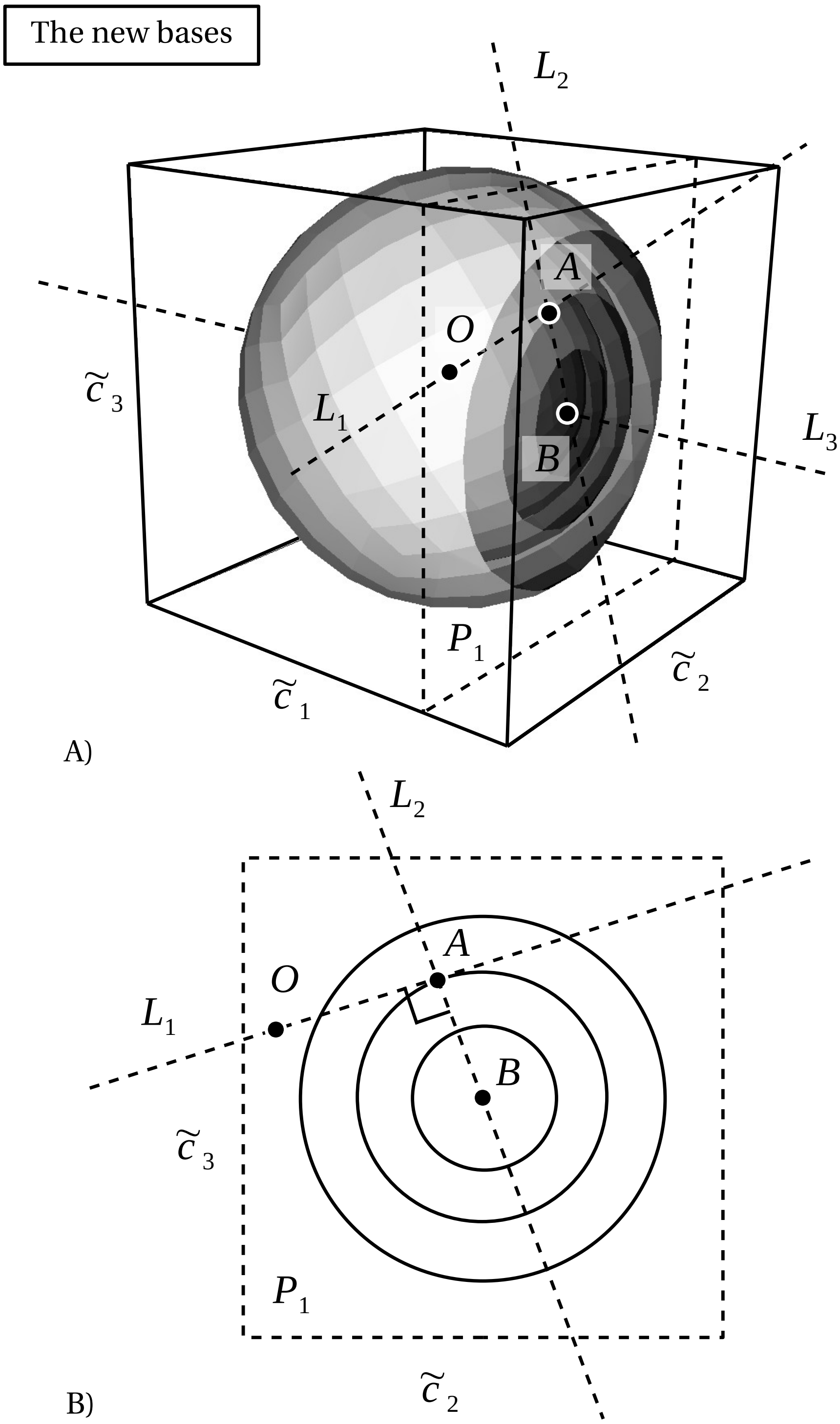

Figure 1.4.12: An illustration to the search procedure of the conjugate gradient algorithm as seen in the bases in which the isoparametric surfaces of the quadratic form are spheres. A) The quadratic form. B) The slice of the quadratic form made by the plane P_1.

basis for the output vector $\boldsymbol{b}$, see equation (1.4.99), and the columns of $\boldsymbol{L}^{-T}$ to be the new basis for the input vector $\boldsymbol{c}$, see equation (1.4.98). Suppose there are two input vectors orthogonal to each other in the new bases:

$$\tilde{\boldsymbol{c}}_1^T \tilde{\boldsymbol{c}}_2 = 0.$$

In this case we apply the transformation rule (1.4.26) with $\tilde{\boldsymbol{E}}^{-1} = \boldsymbol{L}^T$:

$$\left(\boldsymbol{L}^T \boldsymbol{c}_1\right)^T \left(\boldsymbol{L}^T \boldsymbol{c}_2\right) = 0.$$

We can rewrite the last equation as

$$\boldsymbol{c}_1^T \left(\boldsymbol{L}\boldsymbol{L}^T\right) \boldsymbol{c}_2 = 0,$$

and then by observing (1.4.108) as

$$\boldsymbol{c}_1^T \boldsymbol{A} \boldsymbol{c}_2 = 0.$$

Any two column vectors that satisfy the last equation are called $\boldsymbol{A}$-conjugate. Two $\boldsymbol{A}$-conjugate vectors are orthogonal to each other in the bases that make the isoparametric surfaces of the quadratic form spheres. Therefore, if we wish two vectors to be orthogonal in the new bases, we must make them $\boldsymbol{A}$-conjugate in the old bases. Note, that this holds only for vectors that transform as the input vectors ($\tilde{\boldsymbol{E}} = \boldsymbol{L}^{-T}$).

Next, let us consider the second pillar of the conjugate gradient algorithm: the search procedure in the new bases. The steepest descent algorithm discussed in the two preceding sections tends to revisit the same search directions multiple times, see Figure 1.4.9 B), for example. This is not optimal. We would like to construct the search procedure of the conjugate gradient algorithm such that search directions are not revisited. Consider a quadratic form $F(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$ in the new bases, Figure 1.4.12 A). The isoparametric surfaces of this quadratic form are concentric spheres. The minimum if the quadratic form is at the center of the spheres. Assume we start searching for the minimum at the point O, chosen arbitrary. We choose to make our first step along an arbitrary line L_1. Next, we find the point on this line, the point A in Figure 1.4.12 A), at which the quadratic form is at its minimum. If we draw another arbitrary line, L_2, through the point A such that it is perpendicular to the line L_1, it will unavoidably pass through the point, B, which minimizes the quadratic form on the plane P_1 spanned by the lines L_1 and L_2. This fact can be better observed in Figure 1.4.12 B). As soon as the point A minimizes the quadratic form on the line L_1, the line L_1 at point A is tangential to one of the isoparametric curves on P_1. As the isoparametric curve is a circle, the line orthogonal to any tangential line unavoidably passes through the center of the circle, i.e., through the minimum of the quadratic form on the plane P_1. The last implies that the point B that minimizes the quadratic form on the plane P_1 is on the line L_2. Note, that there are infinitely many lines that are orthogonal to L_1 at point A. Our observation, however, will remain the same if we choose any of these lines: the chosen line will pass through the point that minimizes the quadratic form on the plain spanned by the chosen line and L_1. It will be another point B and another plain P_1, but the observation, in general, will remain the same.

Let us go back to Figure 1.4.12 A) and draw line L_3 through the point B such that it is perpendicular to the plane P_1. It is easy to guess that L_3 will pass through the center of the spheres. Indeed, as point B minimizes the quadratic form on the plane P_1, the plane P_1 is tangential to one of the isoparametric surfaces at point B. As soon as this isoparametric surface is a sphere, the line orthogonal to any tangential plane passes through the center of the sphere if it contains the point at which the sphere and the tangential plane intersect. The last implies that the point that minimizes the quadratic form, i.e., the center of the sphere, is on the line L_3. Therefore, it only remains to find the point that minimizes the quadratic form along the line L_3. This point will be the global minimum of the quadratic form. Note, that we have reached the minimum in just three steps. Of course, another time we may get lucky and construct the plane P_1 such that it contains the center of the spheres. In this case we will be able to reach the minimum of the quadratic form in just two steps. On a really lucky day we may choose L_1 such that it contains the center of the spheres. In this case we will reach the minimum of the quadratic form in just one step. In general, we may conclude that if we search for the minimum of a quadratic form like that, we will be able to find it in at most three steps. Note, this search procedure makes sense only if the isoparametric surfaces of the quadratic form are spheres. Finally, we assume that this search procedure can be extended to a general m-dimensional quadratic form. We just need to keep on construing search lines, L_i, and minimize the functional along them. Each new search line must be orthogonal to all search lines that are already constructed and must contain the minimum of the quadratic form on the last search line. Sooner or later we will not be able to construct a new search line as the amount of constructed search lines equals the amount of dimensions of the space, m. The minimum of the quadratic form on the last search line being transformed into the old bases is the solution we are looking for. In the case of an m-dimensional quadratic form the solution will be reached in at most m steps.

Next, let us derive the conjugate gradient algorithm by incorporating the two pillars discussed above into the steepest descent algorithm. The idea is the following. We iterate in the old (standard) bases such that the iteration steps in the new bases follow the search procedure discussed above. For that we need to construct each new search direction such that it is perpendicular to all previous search directions in the new bases. The notion of $\boldsymbol{A}$-conjugate vectors will help us to do so.

Suppose we have chosen a starting point in the old bases: $\boldsymbol{c}_0$. It transforms into the point O in the new bases, see Figure 1.4.12. We begin the search

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1.5 CALCULUS OF VARIATIONS

1.5.1 VARIATIONAL METHOD

Let us consider a problem of calculating an electric scalar potential induced by two concentric spherical conductors. The inner conductor is a spherical ball. The outer conductor is a spherical shell. The inner conductor is at a potential of Φ_0 Volts. The outer shell is grounded. The space between the outer surface of the ball and the inner surface of the shell is empty. This problem can be solved by integrating the following partial differential equation:

$$\nabla^2\Phi = 0, \tag{1.5.1}$$

and applying proper boundary conditions as it is done in section 6.1.2. There is, however, an alternative approach.

Static systems do not change their state because the potential energy in the static state is minimal. The most illustrative example of such static system is a ball in a parabolic pit. The potential energy due to gravity in this configuration is minimal. All other positions of the ball in the pit correspond to higher potential energies. Consequently, if the ball finds itself in a position displaced from the bottom of the pit, it moves about until it settles down in the bottom of the pit. The ball in the bottom of the pit is the static state of the system. The two spherical conductors that we would like to analyze are somewhat like the ball in the bottom of the pit. That is, the electric charge in the conductors settles in the static state shortly after the inner conductor get attached to the voltage source. The potential energy in the static state is minimal possible. It is stored in the electrostatic field and can be evaluated as

$$F(\Phi) = \frac{\epsilon_0}{2}\iiint_\Omega |\vec{E}|^2\, dV = \frac{\epsilon_0}{2}\iiint_\Omega |\vec{\nabla}\Phi|^2\, dV, \tag{1.5.2}$$

where Ω is all the space in the universe, see, for instance, section 2.4.3 in [22]. As the electrostatic field in this configuration exists only in between the conductors, we can restrict the domain of integration, Ω, to the space between the conductors. The electrostatic potential we are looking for minimizes the potential energy (1.5.2) for given boundary conditions. The boundary conditions in this case are: $\Phi = \Phi_0$ Volts on the inner boundary and $\Phi = 0$ Volts on the outer boundary of Ω, see Figure 6.1.2. Therefore, we can replace the task of direct integration of the partial differential equation (1.5.1) with the task of minimizing the potential energy given by equation (1.5.2). This method of solving problems is called variational method. In the framework of the variational method, the expression being minimized does not necessarily have to be the potential energy. Any expression that is minimized by the solution to the initial partial differential equation will do. In the framework of the variational method such expression is called a functional. Arguments of functionals are independent functions, not independent variables. A functional maps functions to real numbers.

To be able to apply the variational method in the framework of the finite element method, we need to answer the following three questions:

- How to convert a partial differential equation into a functional?
- How to convert a functional into a partial differential equation?
- How to minimize a functional?

The first and the third questions above are quite reasonable. The second question is suspicious at very least. Indeed, why anyone would want to convert the functional back into the partial differential equation just after having converted the partial differential equation into the functional? The right place to answer this question is chapter 4. Here is a brief summary of the answer. The functional encodes not only the partial differential equation. It also encodes some boundary and interface conditions. That is, minimization of the functional not only solves the partial differential equation, but also enforces some boundary and interface conditions. The boundary and interface conditions embedded into the functional are called natural. A boundary or interface condition is essential if it is not natural. Essential boundary and essential interface conditions are not taken care of by the functional (it is not natural for the functional to do so) and need to be enforced elsewhere, see section 4.1.5 for more details. In short, the procedure for converting the functional back to the partial differential equation can be perceived as a some kind of inventorization of boundary and interface conditions, see also [25] and [30]. It helps to sort the boundary and interface conditions in two categories: natural and essential.

Chapter 5 is dedicated to the third question above. We will answer the first and the second questions in sections 1.5.3 and 1.5.4, respectively. But before we do so, we need to consider abstract vector spaces.

1.5.2 Abstract vector spaces

The ordinary vectors were introduced by Oliver Heaviside in the end of the nineteenth century, see section 1.1.1. Since the nineteenth century the notion of a vector has become a bit more abstract . A higher abstract level offers an advantage: it allows reusing the same linear algebra in seemingly different circumstances. That is, all good things that constitute linear algebra, such as vectors, matrices, linear transformations, eigenvalues, etc., can, for example, be used for studying electric signals as well as for studying propagation of quantum mechanical objects as both, electric signals and wave functions, can be perceived as elements of abstract vector spaces. In this section we will consider the definition of abstract vectors. To avoid a confusion we will denote ordinary vectors in two- and three- dimensional Euclidean pace by a Latin letter with an arrow, $\vec{v}$ or $\vec{A}$, a column vector as a lower-case boldface Latin letter, $\boldsymbol{v}$, and an abstract vector as a lower-case boldface Latin letter with an arrow, $\vec{\boldsymbol{v}}$.

An abstract vector is defined as an element of an abstract vector space. The abstract vector space is a collection of elements that satisfy the following rules:

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
- $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
- For every two vectors $\vec{a}$ and $\vec{b}$ there exists a third vector $\vec{c}$ such that $\vec{a}+\vec{c}=\vec{b}$. All three vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, belong to the same vector space. This implies existence of a unique zero vector, $\vec{a}+\vec{0}=\vec{a}$.
- $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$
- $(\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a}$
- $\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}$
- $1\vec{a}=\vec{a}$

In these rules, $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{0}$ are abstract vectors, elements of an abstract vector space; α, β, and 1 are scalars. A sum of any two vectors from a vector space must be a valid vector from the same vector space. A scaled version of a vector must belong to the vector space which contains the initial (unscaled) vector.

To enjoy all good things linear algebra has to offer, we need to find a way to convert abstract vectors and their linear transformations into column vectors and matrices, respectively. To this end, we can adapt the conversion procedure we have used on pages 74 - 79 to convert the multidimensional vectors and linear transformations into the column vectors and matrices. This is an easy task as the multidimensional vectors discussed on pages 74 - 79 are, essentially, a specific example of abstract vectors. The space of multidimensional vectors, $\mathbb{R}^m$, is spanned by m basis vectors. An abstract vector space, in general, can have infinite amount of dimensions. Therefore, to generalize the discussion on pages 74 - 79 to the abstract vector spaces we just need to assume that the bases for input and output vectors, $\{\vec{e}_i\}$ and $\{\vec{g}_i\}$, can contain an infinite amount of basis vectors. Consequently, the column vectors in equations (1.4.11), (1.4.12), and the matrix (1.4.15) can, in general, have infinite amount of components. Furthermore, we have to assume that the implementation of the inner product, $\langle\vec{a},\vec{b}\rangle$, for various types of abstract vectors can differ from (1.4.19) although all implementations of the inner product must yield a scalar and must obey the four rules (1.4.7).

Having a valid abstract vector space and a valid inner product we can chose two bases, convert the abstract vectors and their transformations into column vectors and matrices and exercise linear algebra regardless of what the abstract vectors are, exactly. If we manage to convince ourselves that elephants satisfy the seven rules above and can find a valid inner product of two elephants such that we can linearly decompose an elephant as in (1.4.8), we

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CHAPTER 2

ELECTROSTATICS

2.1 ELECTRIC SCALAR POTENTIAL

2.1.1 PARTIAL DIFFERENTIAL EQUATION

The electric field in static approximation is described by equations (1.3.10). As discussed in section 1.3.2, the electrostatic field is conservative under any circumstances. Consequently, we can derive the electrostatic field from a gradient of the electric scalar potential:

$$\vec{E} = -\vec{\nabla}\Phi. \tag{2.1.1}$$

Then the second equation in (1.3.10) is satisfied automatically as a curl of a gradient of a scalar field always equals zero. Next, we rewrite the first equation in (1.3.10) as

$$\vec{\nabla} \cdot \left(\epsilon \vec{E}\right) = \rho_f.$$

By substituting (2.1.1) into the last equation we arrive into the following partial differential equation:

$$-\vec{\nabla} \cdot \left(\epsilon \vec{\nabla}\Phi\right) = \rho_f. \tag{2.1.2}$$

This equation is an adequate description of the most problems in electrostatics. We assume that the permittivity, ϵ, is a real-valued scalar function of spatial coordinates. The permittivity is discontinuous on interfaces between dissimilar dielectric materials. In the literature on the finite element method equation (2.1.2) is known as div-grad equation.

2.1.2 DIRICHLET, NEUMANN, AND ROBIN BOUNDARY CONDITIONS

Equation (2.1.2) describes a family of solutions. A selection of the unique solution out of the family of solutions requires proper boundary conditions.

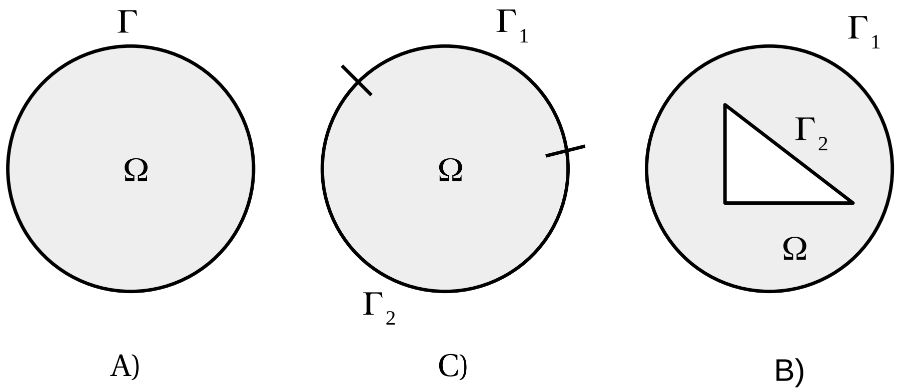

Figure 2.1.1: Examples of simple domains.

There are three different boundary conditions relevant to problems in electrostatics: the Dirichlet boundary condition, the Neumann boundary condition, and the Robin boundary condition. In the following we will prove that the Dirichlet and Robin boundary conditions guarantee the uniqueness of the solution to (2.1.2) and the Neumann boundary condition can only guarantee a solution up to an arbitrary constant.

The Dirichlet boundary condition is considered first. Let us assume that equation (2.1.2) is specified in a simple three-dimensional domain Ω bounded by a closed surface Γ as schematically depicted in Figure 2.1.1 A). The Dirichlet boundary condition specifies the value of electric potential on the boundary of the domain Γ. The partial differential equation (2.1.2) written together with the Dirichlet boundary condition,

$$\begin{aligned} -\vec{\nabla}\cdot\left(\epsilon\vec{\nabla}\Phi\right) = \rho_f \quad &\text{in} \quad \Omega \quad \text{(i)}, \\ \Phi = \eta \quad &\text{on} \quad \Gamma \quad \text{(ii)}, \end{aligned} \tag{2.1.3}$$

constitutes a boundary value problem. Note, that η may vary along the boundary Γ or may simply be a constant. Let us assume that there are two solutions to the boundary value problem (2.1.3), namely Φ_a and Φ_b. Then the difference between them, $\Phi_d = \Phi_a - \Phi_b$, satisfies the following homogeneous boundary value problem:

$$\begin{aligned} \vec{\nabla}\cdot\left(\epsilon\vec{\nabla}\Phi_d\right) = 0 \quad &\text{in} \quad \Omega \quad \text{(i)}, \\ \Phi_d = 0 \quad &\text{on} \quad \Gamma \quad \text{(ii)}. \end{aligned} \tag{2.1.4}$$

Next, let us substitute $\Psi = \Phi_d$ and $\Phi = \Phi_d$ into the first Green's scalar identity (1.1.58):

$$\iiint_\Omega \left[\Phi_d \underbrace{\vec{\nabla}\cdot\left(\epsilon\vec{\nabla}\Phi_d\right)}_{=0} + \left(\epsilon\vec{\nabla}\Phi_d\right)\cdot\vec{\nabla}\Phi_d\right] dV = \oint\!\!\!\oint_\Gamma \left(\Phi_d\epsilon\vec{\nabla}\Phi_d\right)\cdot d\vec{S}.$$

The first term in the last equation equals zero as suggested by the first equation in (2.1.4). Then, after simplifying the rest of the terms, the last equation becomes

$$\iiint_{\Omega} \epsilon \mid \vec{\nabla}\Phi_d \mid^2 dV = \oiint_{\Gamma} \Phi_d \left(\epsilon\vec{\nabla}\Phi_d\right) \cdot d\vec{S}. \tag{2.1.5}$$

According to (ii) in (2.1.4), Φ_d equals zero on Γ. This forces the integral on the right-hand side of (2.1.5) to zero implying that

$$\iiint_{\Omega} \epsilon \mid \vec{\nabla}\Phi_d \mid^2 dV = 0. \tag{2.1.6}$$

The integrand in the last equation is non-negative. Therefore, the integral in the last equation equals zero only if $\vec{\nabla}\Phi_d = 0$ at all points of the domain. The last means that Φ_d equals some constant Φ_0. On the other hand, $\Phi_d = 0$ on Γ, i.e., (ii) in (2.1.4), meaning that $\Phi_0 = 0$. Thus, $\Phi_d = 0$ at all points of the domain and $\Phi_a = \Phi_b$ at all points of the domain. The last implies that the Dirichlet boundary condition specified on the boundary Γ of the simple domain shown in Figure 2.1.1 A) ensures uniqueness of the solution to (2.1.2). Similar prove can be made for any other domain with multiple boundaries. Examples of two such domains are shown in Figures 2.1.1 B) and C). To prove the solution uniqueness in a domain with multiple boundaries, the integral on the right-hand side of (2.1.5) must be written separately for each boundary:

$$\oiint_{\Gamma} \Phi_d \left(\epsilon\vec{\nabla}\Phi_d\right) \cdot d\vec{S} = \iint_{\Gamma_1} \Phi_d \left(\epsilon\vec{\nabla}\Phi_d\right) \cdot d\vec{S} + \iint_{\Gamma_2} \Phi_d \left(\epsilon\vec{\nabla}\Phi_d\right) \cdot d\vec{S} + \ldots \tag{2.1.7}$$

Then each integral on the right-hand side of the last equation is driven to zero by specifying Dirichlet boundary conditions on the corresponding boundary. Consequently, equation (2.1.6), the discussion immediately below it, and the conclusion remain the same.

Next, let us consider the Neumann boundary condition applied on the boundary Γ of the domain Ω depicted in Figure 2.1.1 A). The corresponding boundary value problem can be specified as the following:

$$\begin{aligned} -\vec{\nabla} \cdot \left(\epsilon\vec{\nabla}\Phi\right) &= \rho_f \quad \text{in} \quad \Omega \quad \text{(i)}, \\ \epsilon\hat{n} \cdot \vec{\nabla}\Phi &= \sigma \quad \text{on} \quad \Gamma \quad \text{(ii)}. \end{aligned} \tag{2.1.8}$$

Similarly to the case of the Dirichlet boundary condition, we assume that there are two different solutions to (2.1.8), Φ_a and Φ_b. Then the difference $\Phi_d = \Phi_a - \Phi_b$ satisfies the following homogeneous boundary value problem:

$$\begin{aligned} \vec{\nabla} \cdot \left(\epsilon\vec{\nabla}\Phi_d\right) &= 0 \quad \text{in} \quad \Omega \quad \text{(i)}, \\ \epsilon\hat{n} \cdot \vec{\nabla}\Phi_d &= 0 \quad \text{on} \quad \Gamma \quad \text{(ii)}. \end{aligned}$$

Equation (2.1.5) holds for this boundary value problem as well. The Neumann boundary condition drives the integral on the right-hand side of (2.1.5) to zero. To see this, we separate the vector normal to the boundary, $\hat{n}$, from $d\vec{S}$:

$$d\vec{S} = \hat{n}dS.$$

… Some pages have been omitted …

Box 2.1.1: Static scalar boundary value problem

The electrostatic phenomena are described by the following static scalar boundary value problem:

$$\begin{aligned}
-\vec{\nabla}\cdot\left(\epsilon\vec{\nabla}\Phi\right) = \rho_f &\quad \text{in} \quad \Omega \quad \text{(i)},\\
\Phi = \eta &\quad \text{on} \quad \Gamma_{Dn} \quad \text{(ii)},\\
\epsilon\hat{n}\cdot\vec{\nabla}\Phi + \gamma\Phi = \sigma &\quad \text{on} \quad \Gamma_{Rm} \quad \text{(iii)},\\
\Phi_+ = \Phi_- &\quad \text{on} \quad \Gamma_{Ik} \quad \text{(iv)},\\
\epsilon_+\hat{n}\cdot\vec{\nabla}\Phi_+ - \epsilon_-\hat{n}\cdot\vec{\nabla}\Phi_- = -\kappa_f &\quad \text{on} \quad \Gamma_{Ik} \quad \text{(v)}.
\end{aligned} \tag{2.1.20}$$

- The Robin boundary condition (iii) becomes the Neumann boundary condition for $\gamma = 0$. In the case of the Robin boundary condition:
$$\gamma > 0 \quad \text{on} \quad \Gamma_{Rm}. \tag{2.1.21}$$

- Specification of the Dirichlet (ii) or the Robin (iii) boundary condition on one of the boundaries ensures the uniqueness of the solution.

- The Neumann boundary condition, i.e., (iii) with $\gamma = 0$, ensures the uniqueness of the solution with a precision to a constant. However, if the Dirichlet or Robin boundary condition with γ as in (2.1.21) is specified on at least one other boundary, the solution will be unique.

- The presence of the interfaces between the materials, Γ_{Ik}, does not affect the uniqueness of the solution in any way. Equations (iv) and (v) describe the interface conditions.

- The index "+" refers to the space immediately next to the interface in the direction of vector $\hat{n}$. The index "−" refers to the space immediately next to the interface in the direction opposite to $\hat{n}$.

- The term κ_f accommodates the free surface charge on all interfaces between dissimilar materials. The surface charge κ_f is not included in ρ_f. Thinking that $\rho_f = \rho'_f + \kappa_f$ is wrong.

condition must be set on the boundary of a floating conductor:

$$\begin{aligned}
\Phi_+ = \Phi_- &\quad \text{on} \quad \Gamma_{Ik},\\
\epsilon_+\hat{n}\cdot\vec{\nabla}\Phi_+ = \epsilon_-\hat{n}\cdot\vec{\nabla}\Phi_- &\quad \text{on} \quad \Gamma_{Ik},
\end{aligned}$$

where subscripts '-' and '+' refer to the inner and outer spaces of the floating conductor, respectively. The relative permittivity inside the floating conductor can be chosen somewhat arbitrary, but it must have a relatively high value such as $\epsilon_{r-} = 10^6$, see [26]. In comparison, the relative permittivity outside the floating conductor is $\epsilon_{r+} = 1$ for vacuum, $\epsilon_{r+} = 80$ for water, and $\epsilon_{r+} < 15000$ for barium titanate (a ceramic material used in miniature capacitors). As soon as we treat floating conductors as dielectric materials of high permittivity, there is no need to add new entries to the general boundary value problem on their account: they get covered by the interface conditions (iv) and (v) of (2.1.20).

2.3 Two-dimensional problems

By default, all problems in electromagnetics are three-dimensional. As three-dimensional problems require more computer resources than two- dimensional problems, it is always beneficial to recast a three-dimensional problem into a two-dimensional problem. The last is possible if the initial three-dimensional problem exhibits a symmetry. We can distinguish two kinds of symmetries with this respect: the translation symmetry and the rotation symmetry.

Let us consider the translation symmetry first. The translation symmetry arises in problems that describe straight, infinitely long cable-like structures with a cross section that does not change along the structure. All cross sections made by planes perpendicular to the axis of the structure are the same. We can choose one cross section and use it as a two-dimensional planar problem domain.

The rotation symmetry arises in problems that describe bodies of revolution. In such problems we adapt the cylindrical coordinate system, see Figure B.0.1. In doing so, we orient the cylindrical coordinate system such that the z axis coincides with the axis of rotation symmetry. Then the content of all rz half-planes will be identical. We choose one rz half-plane and treat it as a two-dimensional axisymmetric domain.

A good question may arise at this moment. Why have we chosen the cylindrical coordinate system to describe an axisymmetric domain? The spherical coordinate system can be used to exploit a rotation symmetry as well. The answer to this question is the following. The deal.II library assumes that two-dimensional problem domains are described in the Cartesian coordinate system. We can treat the cylindrical coordinates r and z as Cartesian with a bit of extra effort. We cannot do this easily with the spherical coordinates. Let us elaborate on this.

Strictly speaking, the coordinates r and z are not Cartesian coordinates. They are cylindrical coordinates. For instance, if we wish to calculate a divergence, we will have to use equation (B.0.7) with the middle therm discarded:

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_r\right) + \frac{\partial F_z}{\partial z}.$$

Simply using equation (1.1.28) with y discarded and x replaced by r:

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_r}{\partial r} + \frac{\partial F_z}{\partial z} \qquad \text{(wrong)}$$

will not do. However, for the purpose of evaluating the integrals that constitute the numerical recipes we can treat the r and z coordinates as Cartesian. Consider, for example, the recipe on page 333. To be able to evaluate the integrals that constitute the entries of the system matrix, A_{ij}, and the right-hand side, b_i, we need to evaluate functions, function gradients, area elements, and arc length elements. Evaluating functions is straightforward: function evaluation in the rz plane does not differ from function evaluation in the xy plane. The expression for gradient in the two-dimensional Cartesian coordinate system, i.e.,

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}, \tag{2.3.1}$$

is identical to that of the cylindrical coordinate system under the assumption of rotation symmetry:

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{\partial}{\partial z}\hat{z}.$$

The last is, essentially, equation (B.0.9) with the middle term discarded. Due to the fact that the last two equations are identical the gradient in the rz half-plane of the cylindrical coordinate system and the gradient in the Cartesian coordinate system can be evaluated by the same algorithm. Therefore, we can use the code of deal.II library (it expects the two coordinates to be Cartesian x and y) for evaluating the gradients in the rz half-plane with no extra effort: we just need to interpret the coordinates as r and z, not as x and y. The area and the arc length elements are different in the xy and the rz planes. These elements in the rz half-plane, however, can be converted into corresponding Cartesian elements with a little effort. More on this in section 5.6.1. It must be stressed that the gradient in the spherical coordinate system under assumption of rotation symmetry,

$$\vec{\nabla}\Phi = \frac{\partial \Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\theta},$$

(here r is measured from the origin, not from the axis of symmetry) cannot be easily converted to the form of equation (2.3.1). This is, essentially, the reason why we prefer the cylindrical coordinate system over the spherical coordinate system when describing axisymmetric problem domains.

2.4 REFLECTION SYMMETRY

In the preceding section we have discussed how to reduce a three-dimensional problem to a two-dimensional problem by exploiting the translation and rotation symmetries. In this section we will consider how to reduce a size of a

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Chapter 3

Magnetostatics

The magnetostatic part of the Maxwell's equations is given by (1.3.23). Equation (i) in (1.3.23) suggests that magnetostatic field, $\vec{B}$, is always solenoidal. The last is true regardless the source of magnetostatic field. The magnetostatic field induced by free currents curls around current lines. The magnetostatic field due to magnetization curls around bound currents. The auxiliary vector field $\vec{H}$ also curls around free-current lines. Thus, in the most general case, both, the magnetostatic field $\vec{B}$ and the auxiliary field $\vec{H}$ contain solenoidal vector components and cannot be derived from a scalar potential. Therefore, in the most general case, problems in magnetostatics must be formulated in terms of a vector potential. There are, however, exceptions that allow the auxiliary vector field $\vec{H}$ to be derived as a gradient of a magnetic scalar potential as $\vec{H}$ does not curl around bound currents. These exceptions correspond to the first two options listed on page 65. There are two kinds of magnetic scalar potentials: the total magnetic scalar potential and the reduced magnetic scalar potential. In this chapter we will first consider the total and reduced magnetic scalar potentials and then we will consider the vector magnetic potential.

3.1 Total magnetic scalar potential

Suppose that the problem we would like to solve allows us to exclude all free volume currents, $\vec{J}_f$, and all free surface currents, $\vec{K}_f$, from the problem domain. That is,

$$\vec{J}_f = 0 \quad \text{and} \quad \vec{K}_f = 0.$$

This corresponds to the first two options in the list for auxiliary field $\vec{H}$ on page 65. As soon as there is no free currents in the problem domain, the auxiliary vector field $\vec{H}$ is purely conservative. Consequently, we can derive the auxiliary vector field $\vec{H}$ as a gradient of a total magnetic scalar potential:

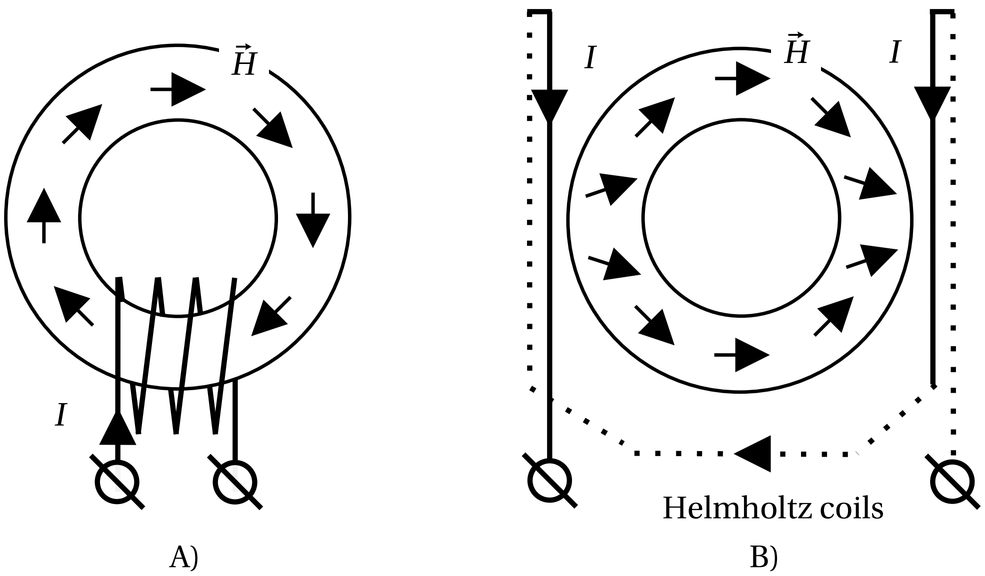

Figure 3.1.1: A) An inductor with a toroidal magnetic core. B) A toroidal magnetic core in Helmholtz coils. A side-view of the Helmholtz coils is shown. The solid current lines of the coils are located in front of the dotted lines.

$$\vec{H} = -\vec{\nabla}\Psi. \tag{3.1.1}$$

There are, however, exceptions. Let us consider the inductor with a toroidal magnetic core depicted in Figure 3.1.1 A). Here, we assume that the problem domain is the interior space of the magnetic core. The problem domain does not contain any free currents. All free currents go around the problem domain. Yet, we cannot apply (3.1.1) as the field lines of $\vec{H}$ are closed. This example, however, does not suggests that (3.1.1) cannot be used in all problems defined in multiply-connected domains[1]. The configuration shown in Figure 3.1.1 B) illustrates this very point: the magnetic material is shaped as a toroid, but $\vec{H}$ field lines are not closed, so we can use equation (3.1.1) in this case. A simple test can be suggested to spot the problem domains in which equation (3.1.1) cannot be applied: if it is possible to draw an Amperian loop within the problem domain such that non-zero free current crosses the open surface spanned by the loop, then equation (3.1.1) cannot be applied. All the points of the loop must lie within the problem domain. Figure 3.1.2 illustrates application of this test to the problem domain depicted in Figure 3.1.1 A). In this case it is possible to draw an Amperian loop within the domain such that non-zero current ($4I$, in this particular case) crosses the surface spanned by the Amperian loop. Therefore, the total magnetic scalar potential cannot be used in this domain. It is not possible to draw such an Amperian loop in the domain shown in Figure 3.1.1 B). Therefore, the total magnetic scalar potential can be applied in this domain. Note, that it is possible to draw an Ampe-

[1] This is just an alternative name for a domain that looks like a doughnut or a pretzel.

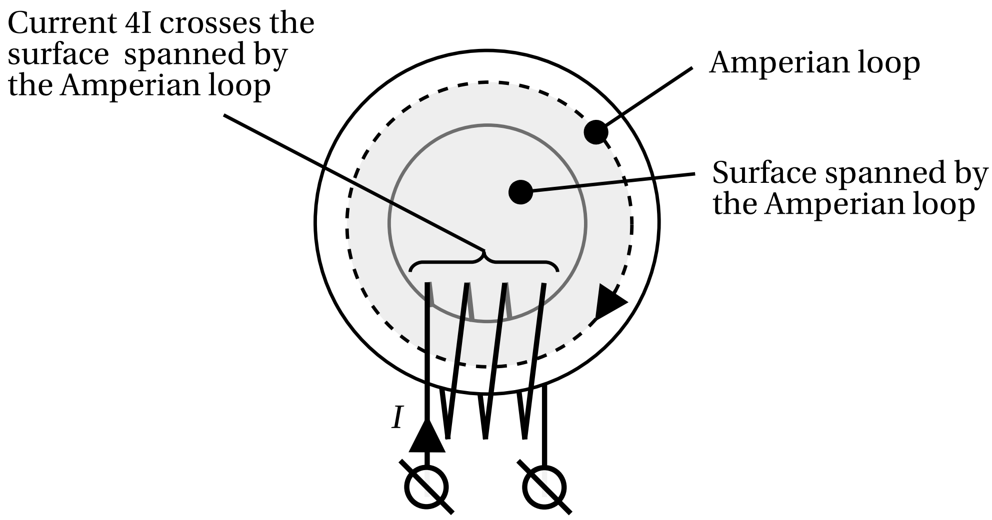

Figure 3.1.2: An Amperian loop drawn within the problem domain depicted in Figure 3.1.1 A). The total current of $4I \neq 0$ crosses the surface spanned by the Amperian loop. Therefore, the total magnetic scalar potential cannot be applied in this problem domain.

rian loop within the toroid shown in Figure 3.1.1 B) and attach to it a surface of a very exotic shape such that some current lines will cross the surface. A careful consideration in this case, however, will reveal that the total current crossing the surface is zero as the amount of current lines that cross the surface in one direction equals the amount of current lines that cross the surface in the opposite direction, so the net current will be identical to zero and the main conclusion, i.e., the applicability of (3.1.1), will remain the same.

As discussed above, we assume that there are no free currents in the problem domain. A good question is: what induces the magnetic field, exactly, if there are no free currents? There are two answers. The first is - free currents outside the problem domain. The information about the free currents outside the problem domain is coupled to the problem domain implicitely via boundary conditions. The second answer is - permanent magnets. In the discussion below we will describe the permanent magnets by means of scaled magnetic volume and surface charge densities. That is, we will treat the permanent magnets as field sources. For this reason, we need to add the magnetization of the permanent magnets into consideration. Let us do this next.

We split all magnetic materials on hard magnetic materials and soft magnetic materials. There is no well-defined boundary between hard and soft magnetic materials. The magnetic materials are sorted in these two categories based on the value of their coercivity. Magnetic materials with high coercivity are categorized as hard. Neodymium magnets, for instance, are made of hard magnetic material. Magnetic materials with low coercivity are

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Box 3.3.1: Static vector boundary value problem

The magnetostatic phenomena are described by the following static vector boundary value problem:

$$
\begin{aligned}
\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A}\right) &= \vec{J}_f \quad &&\text{in} \quad \Omega &&\text{(i)},\\
\hat{n} \times \vec{A} &= \hat{n} \times \vec{G} \quad &&\text{on} \quad \Gamma_{Dn} &&\text{(ii)},\\
\frac{1}{\mu} \hat{n} \times \left(\vec{\nabla} \times \vec{A}\right) + \gamma \hat{n} \times \left(\hat{n} \times \vec{A}\right) &= \vec{Q} \quad &&\text{on} \quad \Gamma_{Rm} &&\text{(iii)},\\
\hat{n} \times \vec{A}_+ &= \hat{n} \times \vec{A}_- \quad &&\text{on} \quad \Gamma_{Ik} &&\text{(iv)},\\
\frac{1}{\mu_+} \hat{n} \times \left(\vec{\nabla} \times \vec{A}_+\right) - \frac{1}{\mu_-} \hat{n} \times \left(\vec{\nabla} \times \vec{A}_-\right) &= \vec{K}_f \quad &&\text{on} \quad \Gamma_{Ik} &&\text{(v)}.
\end{aligned}
\tag{3.3.31}
$$

- No gauge is applied to the vector potential. The boundary value problem has many solutions. It, however, yields a unique solution for the curl of the vector potential and, thus, a unique solution for the magnetic field $\vec{B}$.
- The Robin boundary condition (iii) becomes the Neumann boundary condition for $\gamma = 0$. In the case of the Robin boundary condition
$$\gamma > 0.$$
- Specification of a Dirichlet (ii) or Neumann (iii) or Robin (iii) boundary condition on one of the boundaries ensures the uniqueness of the curl of the vector potential.
- The presence of the interfaces between dissimilar materials, Γ_{Ik}, does not affect the uniqueness of the solution in any way. Equations (iv) and (v) describe the interface conditions.
- The index "+" refers to the space immediately next to the interface in the direction of vector $\hat{n}$. The index "−" refers to the space immediately next to the interface in the direction opposite to $\hat{n}$.
- The vector field $\vec{K}_f$ accommodates the surface free-current density on interfaces. The volume free-current density, $\vec{J}_f$, does not contain $\vec{K}_f$. Thinking that $\vec{J}_f = \vec{J}'_f + \vec{K}_f$ is wrong.

3.3.1 C). The results above can be extended to a problem domain with an arbitrary amount of boundaries and interfaces.

Note, that the identity (3.3.29) holds due to the fact that we have replaced the initial interface condition dictated by the Maxwell's equations, i.e., (i) in (3.3.15), with a more restrictive interface condition (i) in (3.3.17). Without this replacement it would be somewhat difficult to drive the integral I_3 in (3.3.20) to zero. Consequently, the integral I_3 would have precipitated on the right-hand side of equation (3.3.30) disturbing the uniqueness of the curl of the solution.

The Box 3.3.1 summarizes the static vector boundary value problem. It is, essentially, the boundary value problem (3.3.18) with a few modifications. First of all, we allow for an arbitrary amount of boundaries and interfaces. The indexes n, m, and k index them. Second, we use plus and minus sign in the subscripts instead of integer numbers to label the spaces on the opposite sides of the interfaces.

3.4 Magnetic vector potential in two dimensions

Three-dimensional simulations are demanding in terms of computational resources. This is even more so if a problem is formulated in terms of a vector potential. For this reason a reduction of a three-dimensional problem to two dimensions is always desirable. It is only possible to reduce a three-dimensional problem to two dimensions if it possesses a symmetry. There are two relevant types of symmetry: translation and rotation. On the other hand, the magnetic vector potential in two dimensions can be either an out-of-plane vector or an in-plane vector, see the discussion on page 8. Therefore, there are four possible combinations of symmetries and types of two-dimensional vectors. They are summarized in Table 3.4.1. We will consider these types of problems one-by-one in the four following sections.

3.4.1 Scalar planar problem

A three-dimensional problem that exhibits translation symmetry gives rise to a two-dimensional scalar planar problem if the two-dimensional vector potential is an out-of-plane vector, see Table 3.4.1. Let us consider a straight, infinitely-long cable-like structure aligned with the z axis of a Cartesian coordinate system in which the vector potential, $\vec{A}$, is oriented parallel to the z axis:

$$\vec{A} = 0\hat{\imath} + 0\hat{\jmath} + A(x, y)\hat{k}. \tag{3.4.1}$$

We also assume that the permeability exhibits the translation symmetry as well:

$$\mu = \mu(x, y). \tag{3.4.2}$$

Then the volume free-current density, $\vec{J}_f$, has to have the same form:

$$\vec{J}_f = 0\hat{\imath} + 0\hat{\jmath} + J_f(x, y)\hat{k}. \tag{3.4.3}$$

	Translation	Rotation
Out-of-plane $\vec{A}$	Scalar planar problem A	Scalar axisymmetric problem A'
In-plane $\vec{A}$	Vector planar problem $\vec{A}$	Does not exist

Table 3.4.1: Two-dimensional problems formulated in terms of magnetic vector potential.

We can establish this fact by substituting (3.4.1) and (3.4.2) into left-hand side of (i) in (3.3.31), differentiating as prescribed by equation (1.1.29), and discovering that the current density on the right-hand side must be in the form given by equation (3.4.3). We reduce such a three-dimensional problem to two dimensions by selecting one of the cross sections made by a plane perpendicular to the z axis as a problem domain.

As soon as we are looking for a solution in a form of (3.4.1), the following holds:

$$\vec{\nabla} \cdot \vec{A} = 0,$$

see equation (1.1.28). The last identity is the Coulomb gauge. Therefore, we can conclude that the vector potential that exhibits this type of translation symmetry is implicitly gauged. The last implies that we can expect a unique solution in terms of the vector potential.

In the current configuration the vector potential (3.4.1) is an out-of-plane vector, see discussion on page 8. Therefore, the magnetic field vector is an in-plane vector. We can tailor the definition of the magnetic vector potential (3.3.1) to the current configuration as

$$\vec{B} = \vec{\nabla} \overset{V}{\times} A. \tag{3.4.4}$$

First, let us adapt the partial differential equation, i.e., (i) in (3.3.31), to current configuration. We can express the curl of the vector potential given by (3.4.1) as

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & 0 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & A \end{vmatrix} = \frac{\partial A}{\partial y}\hat{\imath} - \frac{\partial A}{\partial x}\hat{\jmath}. \tag{3.4.5}$$

… Some pages have been omitted …

Chapter 4

Variational formulations

4.1 Static scalar boundary value problem

4.1.1 Homogeneous boundary value problem

Let us consider the following boundary value problem:

$$\begin{aligned} -\vec{\nabla}\cdot\left(\epsilon\vec{\nabla}\Phi\right) = \rho_f \quad &\text{in} \quad \Omega \quad \text{(i)},\\ \Phi = 0 \quad &\text{on} \quad \Gamma_D \quad \text{(ii)},\\ \epsilon\hat{n}\cdot\vec{\nabla}\Phi + \gamma\Phi = 0 \quad &\text{on} \quad \Gamma_R \quad \text{(iii)}. \end{aligned} \tag{4.1.1}$$

We assume that this boundary value problem is defined on the simple domain depicted in Figure 4.1.1 A). The results of the discussion in this section, however, will not change if the problem domain is configured differently. Figure 4.1.1 B) provides an example of an alternative configuration of the problem domain. The Dirichlet and Robin boundary conditions are specified on the boundaries Γ_D and Γ_R, respectively. It is assumed that

$$\gamma > 0 \quad \text{on} \quad \Gamma_R \tag{4.1.2}$$

for the Robin boundary condition. The Robin boundary condition becomes Neumann boundary condition if $\gamma = 0$.

In this section we perceive the electric scalar potential Φ as an abstract vector, an element of an abstract vector space $\mathcal{V}_0$. An artistic impression of this vector space is shown in Figure 1.5.1 C). We assume that all elements of $\mathcal{V}_0$ satisfy both boundary conditions, (ii) and (iii) in (4.1.1). The inner product in $\mathcal{V}_0$ is defined by equation (1.5.3). Here it is reprinted in a slightly different form:

$$\langle\Phi,\Psi\rangle = \iiint_\Omega \Phi\Psi\, dV, \tag{4.1.3}$$

where Ψ is another element of the abstract vector space $\mathcal{V}_0$. Next, we perceive the partial differential equation as an operation of linear transforma-

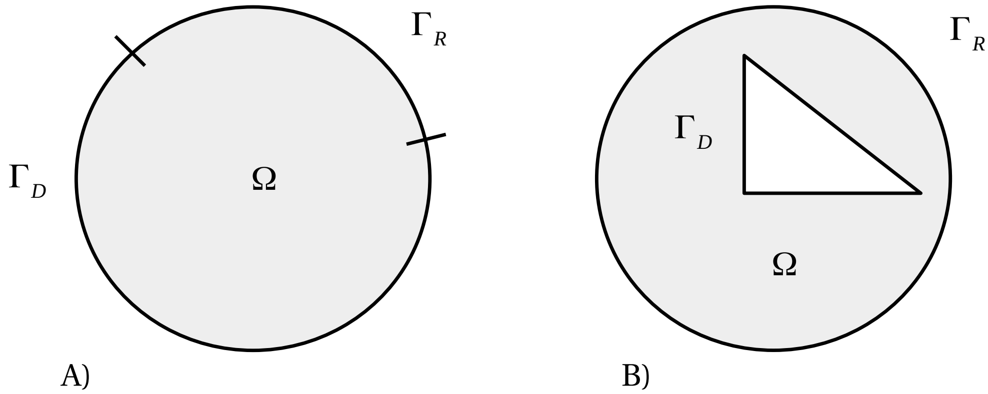

Figure 4.1.1: A) The problem domain on which the boundary value problems (4.1.1), (4.1.16), (4.2.1), and (4.2.19) are defined. B) An alternative configuration of the problem domain. Two different domains are shown to illustrate that the exact configuration of the boundaries is not important for the discussion made in the text.

tion. That is, we express equation (i) in (4.1.1) in terms of a linear transformation:

$$T(\Phi) = \rho_f. \tag{4.1.4}$$

Here by linear transformation T we mean the following composition of differential operators:

$$T(...) = -\vec{\nabla}\cdot\left(\epsilon\vec{\nabla}(...)\right). \tag{4.1.5}$$

Next, let us compose a functional by applying the three-steps procedure described on page 132.

In the fist step of this procedure we need to prove that the linear transformation (4.1.5) is symmetric, i.e., the following identity holds:

$$\langle T(\Phi),\Psi\rangle = \langle\Phi, T(\Psi)\rangle. \tag{4.1.6}$$

To do so, we note that

$$\langle T(\Phi),\Psi\rangle = -\iiint_\Omega \Psi\vec{\nabla}\cdot\left(\epsilon\vec{\nabla}\Phi\right)dV.$$

Application of the second Green's scalar identity (1.1.59) to the last equation yields

$$\langle T(\Phi),\Psi\rangle = \underbrace{-\iiint_\Omega \Phi\vec{\nabla}\cdot\left(\epsilon\vec{\nabla}\Psi\right)dV}_{\langle\Phi,T(\Psi)\rangle}\underbrace{-\oint\!\!\!\oint_\Gamma \epsilon\left(\Psi\vec{\nabla}\Phi-\Phi\vec{\nabla}\Psi\right)\cdot d\vec{S}}_{=0}. \tag{4.1.7}$$

The second integral in the last equation vanish on all boundaries as both, Φ and Ψ, satisfy boundary conditions (ii) and (iii) in (4.1.1). That is, by substituting the boundary conditions (ii) and (iii) in (4.1.1) into the last integral in

(4.1.7) yields

$$\oiint_{\Gamma}\epsilon\Big(\Psi\vec{\nabla}\Phi-\Phi\vec{\nabla}\Psi\Big)\cdot d\vec{S}=\underbrace{\iint_{\Gamma_D}\epsilon\Big(0\vec{\nabla}\Phi-0\vec{\nabla}\Psi\Big)\cdot d\vec{S}}_{I_0}-\iint_{\Gamma_R}\gamma\Big(\Psi\Phi-\Phi\Psi\Big)dS=0. \tag{4.1.8}$$

The first integral on the right-hand side of equation (4.1.7) can be interpreted as $\langle\Phi, T(\Psi)\rangle$ implying that (4.1.6) holds if both, Φ and Ψ, satisfy the boundary conditions (ii) and (iii) in (4.1.1). Therefore, the linear transformation (4.1.5) is symmetric if supplemented with the boundary conditions (ii) and (iii) in (4.1.1).

In the second step of the procedure described on page 132 we need to prove that the linear transformation (4.1.5) is positive definite, i.e., the following condition holds for all elements in $\mathscr{V}_0$

$$\langle\Phi, T(\Phi)\rangle>0. \tag{4.1.9}$$

To do so, we rewrite the first Green's scalar identity (1.1.58) substituting $\Psi=\Phi$:

$$\underbrace{\iiint_{\Omega}\Phi\vec{\nabla}\cdot\Big(\epsilon\vec{\nabla}\Phi\Big)dV}_{-\langle\Phi,T(\Phi)\rangle}+\iiint_{\Omega}\Big(\epsilon\vec{\nabla}\Phi\Big)\cdot\vec{\nabla}\Phi dV=\oiint_{\Gamma}\Big(\Phi\epsilon\vec{\nabla}\Phi\Big)\cdot d\vec{S}. \tag{4.1.10}$$

Let us evaluate the last integral in the last equation on both boundaries, Γ_D and Γ_R:

$$\oiint_{\Gamma}\Big(\Phi\epsilon\vec{\nabla}\Phi\Big)\cdot d\vec{S}=\underbrace{\iint_{\Gamma_D}\Big(0\cdot\epsilon\vec{\nabla}\Phi\Big)\cdot d\vec{S}}_{I_1}-\iint_{\Gamma_R}\gamma\Phi^2dS=-\iint_{\Gamma_R}\gamma\Phi^2dS. \tag{4.1.11}$$

Consequently, the equation (4.1.10) can be rewritten as the following:

$$\langle\Phi, T(\Phi)\rangle=\iiint_{\Omega}\epsilon\Big|\vec{\nabla}\Phi\Big|^2dV+\iint_{\Gamma_R}\gamma\Phi^2dS>0 \tag{4.1.12}$$

implying that (4.1.9) holds for all elements in $\mathscr{V}_0$. Thus, the linear transformation (4.1.5) is positive definite. Note also, that permittivity ϵ is positive at all points of Ω and γ is positive on the boundary Γ_R, i.e., condition (4.2.2). The same conclusion holds in the case of the Neumann boundary condition, i.e., $\gamma=0$.

In the third step of the procedure described on page 132 we derive the functional by constructing a quadratic form as

$$F(\Phi)=\langle\Phi, T(\Phi)\rangle-2\langle\rho_f,\Phi\rangle. \tag{4.1.13}$$

Next, by observing equations (4.1.3) and (4.1.5) we rewrite the functional (4.1.13) in integral form as

$$F(\Phi)=-\iiint_{\Omega}\Phi\vec{\nabla}\cdot\Big(\epsilon\vec{\nabla}\Phi\Big)dV-2\iiint_{\Omega}\Phi\rho_f dV.$$

… Some pages have been omitted …

region of integration, Γ_I:

$$
\begin{aligned}
&-I_5-I_6-I_7=\\
&=-\oiint_{\Gamma_I}\left(\left(\frac{1}{\mu_1}\vec{\nabla}\times\vec{A}_1\right)\times\delta\vec{A}_1\right)\cdot\hat{n}_1 dS-\oiint_{\Gamma_I}\left(\left(\frac{1}{\mu_2}\vec{\nabla}\times\vec{A}_2\right)\times\delta\vec{A}_2\right)\cdot\hat{n}_2 dS-\\
&-\iint_{\Gamma_I}\vec{K}_f\cdot\delta\vec{A}dS=\\
&=-\oiint_{\Gamma_I}\left(\hat{n}_1\times\left(\frac{1}{\mu_1}\vec{\nabla}\times\vec{A}_1\right)\right)\cdot\delta\vec{A}_1 dS-\oiint_{\Gamma_I}\left(\hat{n}_2\times\left(\frac{1}{\mu_2}\vec{\nabla}\times\vec{A}_2\right)\right)\cdot\delta\vec{A}_2 dS-\\
&-\iint_{\Gamma_I}\vec{K}_f\cdot\delta\vec{A}dS=\\
&=\oiint_{\Gamma_I}\underbrace{\left(-\hat{n}\times\left(\frac{1}{\mu_1}\vec{\nabla}\times\vec{A}_1\right)+\hat{n}\times\left(\frac{1}{\mu_2}\vec{\nabla}\times\vec{A}_2\right)-\vec{K}_f\right)}_{=0}\cdot\delta\vec{A}dS=0.
\end{aligned}
$$

Here again we have used the fact that $\hat{n}=\hat{n}_1=-\hat{n}_2$ in Figure 4.2.1 A) and identity (1.1.7). As soon as the variation $\delta\vec{A}$ can take an arbitrary shape on the interface Γ_I we can invoke the fundamental lemma of calculus of variations and deduce that

$$
\frac{1}{\mu_2}\hat{n}\times\left(\vec{\nabla}\times\vec{A}_2\right)-\frac{1}{\mu_1}\hat{n}\times\left(\vec{\nabla}\times\vec{A}_1\right)=\vec{K}_f \quad \text{on} \quad \Gamma_I
$$

what is identical to (v) in (4.2.30).

The integral I_4 in (4.2.40) must equal zero as well. By rearranging the terms of the integral I_4,

$$
\begin{aligned}
I_4&=\oiint_{\Gamma_R}\left(\hat{n}\cdot\left(\frac{1}{\mu_2}\vec{\nabla}\times\vec{A}_2\right)\times\delta\vec{A}_2-\gamma\left(\hat{n}\times\vec{A}_2\right)\cdot\left(\hat{n}\times\delta\vec{A}_2\right)-\vec{Q}\cdot\delta\vec{A}_2\right)dS=\\
&=\oiint_{\Gamma_R}\underbrace{\left(\hat{n}\times\left(\frac{1}{\mu_2}\vec{\nabla}\times\vec{A}_2\right)+\gamma\hat{n}\times\left(\hat{n}\times\vec{A}_2\right)-\vec{Q}\right)}_{=0}\cdot\delta\vec{A}_2 dS=0,
\end{aligned}
$$

and invoking the fundamental lemma of calculus of variations we deduce that

$$
\frac{1}{\mu}\hat{n}\times\left(\vec{\nabla}\times\vec{A}\right)+\gamma\hat{n}\times\left(\hat{n}\times\vec{A}\right)=\vec{Q} \quad \text{on} \quad \Gamma_R \tag{4.2.41}
$$

as $\delta\vec{A}_2$ can take any shape on boundary Γ_R. Equation (4.2.41) is identical to the Robin boundary condition (iii) in (4.2.30).

Therefore, we can conclude that the functional (4.2.36) encodes: the partial differential equation (i) in (4.2.30), the Robin boundary condition (iii) in (4.2.30), and the interface condition (v) in (4.2.30). These are natural boundary and interface conditions. The Dirichlet boundary condition (ii), and the interface condition (iv) in (4.2.30) are essential. They must be imposed elsewhere. In practice, the Dirichlet boundary condition is imposed by constraining the degrees of freedom, and the interface condition (iv) is imposed

Box 4.2.3: Static vector boundary value problem - functional in 3D

The solution to the static vector boundary value problem for the magnetostatic vector potential

$$
\begin{array}{llll}
 & \vec{\nabla}\times\left(\frac{1}{\mu}\vec{\nabla}\times\vec{A}\right)+\eta^2\vec{A}=\vec{J}_f & \text{in}\quad \Omega & \text{(i)},\\
\text{(e)} & \hat{n}\times\vec{A}=\hat{n}\times\vec{G} & \text{on}\quad \Gamma_{Dn} & \text{(ii)},\\
\text{(n)} & \frac{1}{\mu}\hat{n}\times\left(\vec{\nabla}\times\vec{A}\right)+\gamma\hat{n}\times\left(\hat{n}\times\vec{A}\right)=\vec{Q} & \text{on}\quad \Gamma_{Rm} & \text{(iii)},\\
\text{(e)} & \hat{n}\times\vec{A}_+=\hat{n}\times\vec{A}_- & \text{on}\quad \Gamma_{Ik} & \text{(iv)},\\
\text{(n)} & \frac{1}{\mu_+}\hat{n}\times\left(\vec{\nabla}\times\vec{A}_+\right)-\frac{1}{\mu_-}\hat{n}\times\left(\vec{\nabla}\times\vec{A}_-\right)=\vec{K}_f & \text{on}\quad \Gamma_{Ik} & \text{(v)},
\end{array}
\tag{4.2.42}
$$

where

$$
\begin{array}{lll}
\gamma>0 & \text{on}\quad \Gamma_{Rm} & \text{for Robin boundary conditions}\\
\text{and} & & \\
\gamma=0 & \text{on}\quad \Gamma_{Rm} & \text{for Neumann boundary conditions,}
\end{array}
$$

minimizes the following functional:

$$
\begin{aligned}
F(\vec{A}) = &\iiint_\Omega \frac{1}{\mu}\left|\vec{\nabla}\times\vec{A}\right|^2 dV+\sum_m\iint_{\Gamma_{Rm}}\left(\gamma\left|\hat{n}\times\vec{A}\right|^2+2\vec{Q}\cdot\vec{A}\right)dS+\\
&+\eta^2\iiint_\Omega|\vec{A}|^2\,dV-2\underbrace{\iiint_\Omega\vec{J}_f\cdot\vec{A}dV}_{I_J}-2\sum_k\iint_{\Gamma_{Ik}}\vec{K}_f\cdot\vec{A}dS.
\end{aligned}
\tag{4.2.43}
$$

The boundary value problem (4.2.42) is, essentially, the boundary value problem (3.3.31) with the gauging term $\eta^2\vec{A}$ added to the partial differential equation. The last five bullet points below the boundary value problem (3.3.31) can be directly applied to the boundary value problem (4.2.42).

by the choice of the finite elements. The curl-conforming elements such as FE_Nedelec<dim> of deal.II guarantee the continuity of the tangential component of the vector field on their edges and faces.

Finally, we generalize the functional (4.2.36) by allowing the problem domain to have an arbitrary amount of boundaries and interfaces between dissimilar materials. The generalized functional is described in the Box 4.2.3.

Next, let us derive the functional that can be used to solve the two- dimensional planar problems formulated in terms of the magnetic vector potential, $\vec{A}$, see section 3.4.3. Strictly speaking, the cross product and the curl exist only in the three- dimensional space. Therefore, the boundary value problem (4.2.42) and the functional (4.2.43) describe only three- dimensional problems. In section 1.1 we have introduced two- dimensional cross products and curls in order to facilitate the analysis of two- dimensional problems. It is possible to repeat all derivations on pages 238 - 257 using these definitions and deduce the functional that corresponds to the boundary value problem (3.4.61). These derivations, however, would be almost identical to their three-dimensional counterparts on pages 238 - 257. For this reason, we will skip these derivations altogether and invoke the informal procedure described on page 211. This procedure must be extended with the following rule. The volume integrals must be replaced with surface integrals over the two- dimensional domain,

$$\iiint_{\Omega} dV \to \iint_{\Omega} dS,$$

and the surface integrals over the boundaries and interfaces must be replaced with line integrals,

$$\iint_{\Gamma} dS \to \int_{\Gamma} dl.$$

This rule is motivated by the fact that in two-dimensional derivations the Green's identities (1.1.62) and (1.1.63) must be used instead of (1.1.60) and (1.1.61). The functional derived by application of such procedure is presented in the Box 4.2.4. Recall that the boundary value problem (4.2.44) describes planar two-dimensional vector problems. Consequently, the functional (4.2.45) describes planar two-dimensional vector problems as well. The axisymmetric two-dimensional vector problems do not exist, see section 3.4.4.

4.3 PROJECTIONS

Each physical quantity studied in electromagnetics must be modeled by a specific type of finite elements. There are four relevant types of finite elements. We will discuss them in more detail in chapter 5. The projection transformation helps to switch between the finite elements. Consider the following example. Suppose we have run a simulation and got an electric scalar potential Φ as a result. The potential is modeled by the Lagrange finite elements as this type of elements is appropriate for an electric scalar potential. Next, we would like to convert the calculated scalar potential into the electric field, $\vec{E}$. The two physical quantities, Φ and $\vec{E}$, are related as

$$\vec{E} = -\vec{\nabla}\Phi. \tag{4.3.1}$$

The proper finite elements for modeling $\vec{E}$ are the Nedelec finite elements. So, we need to implement the last equation such, that the quantity on the right-hand side, Φ, is modeled by the Lagrange finite elements and the quantity on the left-hand side, $\vec{E}$, is modeled by the Nedelec finite elements, see section 5.4. We can do this with a help of the projection transformation.

… Some pages have been omitted …

Chapter 5

Finite Element method

5.1 Overview of the method

A finite element is a collection of three items [13]: (i) a cell, (ii) a set of shape functions defined on the cell, and (iii) a set of degrees of freedom associated with the shape functions. Let us consider the meaning of these three items and review the terminology associated with them.

In the framework of the finite element method a problem domain is represented by a tessellation. We will call this tessellation a mesh. In the deal.II terminology it is known as triangulation. The deal.II library supports quadrilateral and hexahedral mesh cells. Figure 5.1.1 gives an example of how such cells may look like. The vertices of the cells are often referred to as mesh nodes. The line segments that interconnect them are called edges. In deal.II terminology the edges of a quadrilateral cell and facets of a hexahedral cell are called faces. The deal.II library also supports one-dimensional cells which are, essentially, line segments.

Suppose we have successfully programmed, compiled, and run a finite element program which calculates an electric potential in a problem domain. The output of this hypothetical program will be an ordered collection of real numbers in double format. These numbers are called degrees of freedom. The degrees of freedom, c_j, allow calculating the scalar potential at any point within the problem domain as

$$\Phi(\vec{r}) = \sum_j c_j N_j(\vec{r}). \tag{5.1.1}$$

The functions $N_j(\vec{r})$ in the last equation are called shape functions. They are also known as basis functions or interpolation functions. In other words, within the framework of the finite element method, a solution to a boundary value problem is represented as a sum of scaled shape functions. The degrees of freedom are the scaling coefficients. The shape functions in (5.1.1) are essentially scalar fields that exist locally within a few mesh cells. There

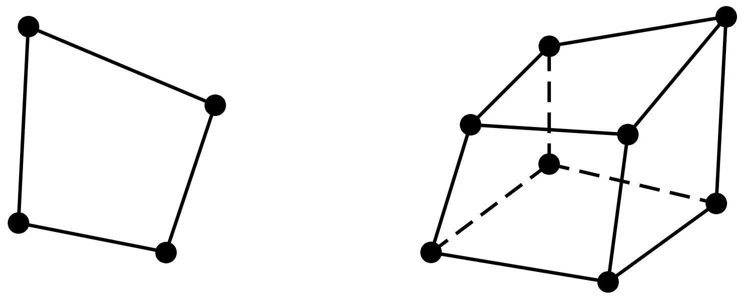

Figure 5.1.1: A) An example of a quadrilateral cell. B) A example of a hexahedral cell.

is another type of shape functions. They are vector fields that exist locally within a few cells. They are used in vector-valued problems. The magnetic vector potential, for example, can be expressed as

$$\vec{A} = \sum_j c_j \vec{N}_j(\vec{r}), \tag{5.1.2}$$

where $\vec{N}_j(\vec{r})$ are vector-valued shape functions. The finite elements that comprise scalar shape functions are called scalar finite elements. Accordingly, the finite elements that comprise vector shape functions are called vector finite elements.

As discussed above, a degree of freedom is just a real number. As any real number, a degree of freedom can be regarded as a functional, C_j, that takes a function as an input and produces a real number as an output. The function in the argument of the functional is a scalar field,

$$c_j = C_j\big(\Phi(\vec{r})\big), \tag{5.1.3}$$

in the case of the scalar finite element. In the case of the vector finite elements the argument of the functional is a vector field,

$$c_j = C_j\big(\vec{A}(\vec{r})\big). \tag{5.1.4}$$

Suppose we wish to approximate a scalar field with shape functions on a particular mesh. To do so, we need to calculate the degrees of freedom, c_j, and then invoke equation (5.1.1). The functional C_j in (5.1.3) can help us to calculate the degrees of freedom. This functional gives instructions on how to convert the scalar field into j-th degree of freedom. These instructions are specific to a particular type of the finite elements. For some finite elements the functional C_j instructs to sample the scalar field at a specific point. For other finite elements it instructs to calculate a particular integral. The interested reader can refer to Chapter 3 in [28] which gives a very accessible overview of common and unusual finite elements and the corresponding degrees of freedom. Equations (5.1.3) and (5.1.4) can be regarded as definitions of the degrees of freedom.

In general, different types of shape functions share common features. Firstly, the support of a shape function is compact. That is, a shape function equals zero outside a small region of the problem domain. There is nothing wrong with functions that have a global support, i.e., exist at all points of the problem domain. It is just the shape functions with compact support do a better job in approximating functions locally. Think, for example, of approximating a function that at a distance looks more like a delta function . It is more convenient to approximate it with shape functions that exist on very small patches, i.e., have compact supports, than with functions that exist at all points of the domain such as harmonics of the Fourier series.

Secondly, shape functions are orthonormal. The notion of orthonormality can be expressed as

$$c_j = C_j(N_i) = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases} \tag{5.1.5}$$

for scalar shape functions and as

$$c_j = C_j(\vec{N}_i) = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases} \tag{5.1.6}$$

for vector shape functions. This is simply a matter of convenience. If the shape functions are orthonormal the changes made by adjusting the degrees of freedom in (5.1.1) or in (5.1.2) are independent of each other. That is, an adjustment of one degree of freedom does not create a necessity to readjust other degrees of freedom. It is easier to find the right combination of the degrees of freedom if the adjustments they make are independent of each other.

Thirdly, the shape functions share some properties with the physical quantities they model so they can take care of some essential interface conditions. For example, in the end of section 4.1.4 and in section 4.1.5 we have discussed that the interface condition (iv) in (2.1.20) is an essential condition. It needs to be imposed by a proper choice of finite elements. The shape functions of the Lagrange finite elements discussed in section 5.2.1 are constructed in such a way that scalar fields approximated by them will be continuous no matter what. That is, the shape functions of Lagrange finite elements guarantee the continuity of the solution. Therefore, the choice of the Lagrange finite elements enforces interface condition (iv) in (2.1.20). Similarly, the choice of the Nedelec finite elements, section 5.3.2, enforces the essential interface condition (iv) in (3.3.31).

The finite element method closely resembles approximation. The most significant difference between the finite element method and approximation is that approximation approximates known functions, while the finite element method approximates unknown functions that are solutions to boundary value problems. Just as it is the case with approximation, the finite ele-

… Some pages have been omitted …

Box 5.5.2: Current vector potential in 2D

The compatibility condition for the curl-curl equation suggests that the free-current density must be derived as a curl of a current vector potential:

$$\vec{J}_f = \vec{\nabla} \overset{V}{\times} T.$$

Then the partial differential equation (i) in (4.2.44) must be replaced with the following equation:

$$\vec{\nabla} \overset{V}{\times} \left(\frac{1}{\mu} \vec{\nabla} \overset{S}{\times} \vec{A} \right) + \eta^2 \vec{A} = \vec{\nabla} \overset{V}{\times} T.$$

Accordingly, the integral

$$I_J = \iint_\Omega \vec{J}_f \cdot \vec{A} dS$$

in the functional (4.2.45) must be replaced with the following two integrals:

$$I_J = \iint_\Omega T\left(\vec{\nabla} \overset{S}{\times} \vec{A}\right) dS - \oint_\Gamma T\left(\hat{n} \overset{S}{\times} \vec{A}\right) dl. \tag{5.5.11}$$

The current vector potential itself can be calculated by solving the following boundary value problem

$$\begin{aligned} -\vec{\nabla} \cdot \left(\vec{\nabla} T\right) &= \vec{\nabla} \overset{S}{\times} \vec{J}_f \quad \text{in} \quad \Omega \quad \text{(i)}, \\ T &= G \quad \text{on} \quad \Gamma_{Dn} \quad \text{(ii)}, \\ \hat{n} \cdot \left(\vec{\nabla} T\right) &= Q \quad \text{on} \quad \Gamma_{Rm} \quad \text{(iii)}. \end{aligned} \tag{5.5.12}$$

This boundary value problem is a modified version of (2.1.20). Note, that in most of the cases only one boundary condition needs to be implemented, either the Dirichlet boundary condition (ii) or the Neumann boundary condition (iii).

5.6 NUMERICAL RECIPES

In this section we will convert the functionals that has been derived in the preceding sections into numerical recipes that can be programmed into a computer. To be precise, we need to replace the integrals in all functionals with quadratures. We, however, will not do so and leave the integrals as they are. I think it is easier to read a recipe in this form. If integrals are replaced with quadrature sums, one needs to track which sum sums over a quadrature and which sum sums over interfaces and boundaries. Furthermore, the information on the types of the integrals (volume, surface, or line) will be lost if the integrals are replaced by quadrature sums in the recipes. This information helps to place the integrals in the right loop in the code. Instead of replacing the integrals with quadratures in all recipes we will just keep in mind that an integral, volume, surface, or line, no matter, must be replaced with the following quadrature sum:

$$I \approx \sum_i L(\vec{q}_i)\left|\det(\boldsymbol{J}_B)\right| s_i,$$

where $L(\vec{q}_i)$ is the integrand evaluated at a quadrature point $\vec{q}_i$, $\left|\det(\boldsymbol{J}_B)\right|$ is the Jacobian evaluated at $\vec{q}_i$, and s_i is the weight coefficient. The product $\left|\det(\boldsymbol{J}_B)\right| s_i$ is what you get when you ask deal.II to "update_JxW_values".

All integrals in the numerical recipes have tags such as I_{a1}. Each component of the system matrix (5.6.7), for example, consists of two integrals, I_{a1} and I_{a2}. These tags are used in the help of the computer code as references. Note that the template mechanism of deal.II makes the computer code invariant to the dimensionality of the integrals. That is, the same peace of code implements the integral I_{a1} in (5.6.7) and I_{a1} in (5.6.14). For this reason, these two different integrals (one is volume integral, another is a surface integral) have the same label, I_{a1}.

In the discussion below we assume that the mesh is matched with interfaces between dissimilar materials. That is, all interfaces are approximated by the faces of the mesh cells such that no interface cuts through a mesh cell.

5.6.1 Static scalar solver (div-grad)

Static scalar solver in three dimensions

The functional for the static scalar boundary value problem is given by (4.1.45). It is reprinted here for convenience:

$$\begin{aligned} F(\Phi) = &\iiint_\Omega \epsilon \mid \vec{\nabla}\Phi \mid^2 dV + \sum_m \iint_{\Gamma_{Rm}} \left(\gamma\Phi^2 - 2\sigma\Phi\right) dS - \\ &-2\sum_k \iint_{\Gamma_{Ik}} \kappa_f \Phi dS - 2\iiint_\Omega \rho_f \Phi dV. \end{aligned} \tag{5.6.1}$$

We convert this functional into a multivariate function by substituting (5.1.1) into it,

$$\begin{aligned} &F(\boldsymbol{c}) = \\ &= \iiint_\Omega \epsilon\left(\vec{\nabla}\sum_j c_j N_j\right)\cdot\left(\vec{\nabla}\sum_j c_j N_j\right) dV + \sum_m \iint_{\Gamma_{Rm}} \gamma\left(\sum_j c_j N_j\right)\cdot\left(\sum_j c_j N_j\right) dS - \\ &-2\sum_m \iint_{\Gamma_{Rm}} \sigma\left(\sum_j c_j N_j\right) dS - 2\sum_k \iint_{\Gamma_{Ik}} \kappa_f\left(\sum_j c_j N_j\right) dS - 2\iiint_\Omega \rho_f\left(\sum_j c_j N_j\right) dV. \end{aligned}$$

In order to find the stationary point of this function, we differentiate it with respect to c_i and equate the result to zero:

$$\frac{\partial F(\boldsymbol{c})}{\partial c_i} = \iiint_\Omega \epsilon \vec{\nabla} N_i \cdot \left(\sum_j c_j \vec{\nabla} N_j \right) dV + \iiint_\Omega \epsilon \left(\sum_j c_j \vec{\nabla} N_j \right) \cdot \vec{\nabla} N_i \, dV +$$
$$+ \sum_m \iint_{\Gamma_{Rm}} \gamma N_i \left(\sum_j c_j N_j \right) dS + \sum_m \iint_{\Gamma_{Rm}} \gamma \left(\sum_j c_j N_j \right) N_i \, dS -$$
$$- 2 \sum_m \iint_{\Gamma_{Rm}} \sigma N_i \, dS - 2 \sum_k \iint_{\Gamma_{Ik}} \kappa_f N_i \, dS - 2 \iiint_\Omega \rho_f N_i \, dV = 0.$$

Next, let us divide the last equation by two and rearrange the terms as the following:

$$\sum_j c_j \left(\iiint_\Omega \epsilon \left(\vec{\nabla} N_i \right) \cdot \left(\vec{\nabla} N_j \right) dV + \sum_m \iint_{\Gamma_{Rm}} \gamma N_i N_j \, dS \right) =$$
$$= \sum_m \iint_{\Gamma_{Rm}} \sigma N_i \, dS + \sum_k \iint_{\Gamma_{Ik}} \kappa_f N_i \, dS + \iiint_\Omega \rho_f N_i \, dV. \tag{5.6.2}$$

Equation (5.6.2) is the i-th equation of the final system of linear equations. By introducing the following notations:

$$A_{ij} = \iiint_\Omega \epsilon \left(\vec{\nabla} N_i \right) \cdot \left(\vec{\nabla} N_j \right) dV + \sum_m \iint_{\Gamma_{Rm}} \gamma N_i N_j \, dS, \tag{5.6.3}$$

and

$$b_i = \sum_m \iint_{\Gamma_{Rm}} \sigma N_i \, dS + \sum_k \iint_{\Gamma_{Ik}} \kappa_f N_i \, dS + \iiint_\Omega \rho_f N_i \, dV, \tag{5.6.4}$$

the system of linear equations implied by (5.6.2) can be written as

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{b}. \tag{5.6.5}$$

This system can be solved for $\boldsymbol{c}$, a column vector filled with degrees of freedom. Then the potential at any point within the problem domain Ω can be calculated by invoking (5.1.1). Finally, we summarize the recipe for the static scalar solver in the Box 5.6.1.

Static scalar solver in two dimensions (planar)

In this section we will adapt the recipe discussed in the preceding section to two-dimensional planar problems. We assume that the initial three- dimensional problem from which the two-dimensional planar problem is derived is described in a Cartesian coordinate system. We also assume that the problem exhibits a translation symmetry along the z axis. That is, any cross section made by a plane perpendicular to the z axis presents the same

Box 5.6.1: Recipe for static scalar solver in 3D

A three-dimensional static scalar boundary value problem given by equation (4.1.44) is solved by minimizing the following functional, i.e., equation (4.1.45):

$$F(\Phi) = \iiint_\Omega \epsilon \,|\, \vec{\nabla}\Phi \,|^2 \, dV + \sum_m \iint_{\Gamma_{Rm}} \left(\gamma\Phi^2 - 2\sigma\Phi\right) dS - \\ -2\sum_k \iint_{\Gamma_{Ik}} \kappa_f \Phi dS - 2\iiint_\Omega \rho_f \Phi dV. \tag{5.6.6}$$

The minimum of a discrete version of this functional is calculated by solving the following system of linear equations:

$$\boldsymbol{Ac} = \boldsymbol{b}.$$

The components of the system matrix and that of the right-hand side are calculated as

$$A_{ij} = \underbrace{\iiint_\Omega \epsilon\left(\vec{\nabla} N_i\right)\cdot\left(\vec{\nabla} N_j\right) dV}_{I_{a1}} + \sum_m \underbrace{\iint_{\Gamma_{Rm}} \gamma N_i N_j dS}_{I_{a2}} \tag{5.6.7}$$

and

$$b_i = \sum_m \underbrace{\iint_{\Gamma_{Rm}} \sigma N_i dS}_{I_{b1}} + \sum_k \underbrace{\iint_{\Gamma_{Ik}} \kappa_f N_i dS}_{I_{b2}} + \underbrace{\iiint_\Omega \rho_f N_i dV}_{I_{b3}}.$$

two-dimensional picture. We choose one of the cross sections to be the two-dimensional planar domain. Next, we expand the volume element of the initial three-dimensional problem domain as

$$dV = dz dS, \tag{5.6.8}$$

where dS is the surface element of the two-dimensional planar domain. Similarly, we expand the surface element of the initial three-dimensional problem domain as

$$dS = dz dl, \tag{5.6.9}$$

where dl denotes the arc length element of the cross section of a three- dimensional surface that represents a boundary or an interface between dissimilar materials. As we have assumed a translation symmetry in the initial three- dimensional problem domain, all integrands in the functional (5.6.6) are independent of the z coordinate. Consequently, we can calculate all the integrals over the z coordinate in advance by analytical methods. Taking into

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CHAPTER 6

LIBRARY OF CLOSED-FORM ANALYTICAL SOLUTIONS

Software verification is an integral part of scientific computing. Ones the recipes for solvers and projectors discussed in the preceding chapter are implemented in a computer code, it is only natural to ask whether the code does what it is supposed to do. The method of exact solutions can help to answer this question. According to this method, one needs to choose a textbook problem with a known closed-form analytical solution, solve this problem numerically with a help of the computer code, and compare the numerical result to the solution from the textbook. The numerical solution must be close to the textbook solution. Moreover, the error of approximation of the textbook solution by the numerical solution should decrease with a decreasing size of mesh cells.

In practice, however, it is quite difficult to find one textbook problem that can cover all components of a boundary value problem and all possible configurations of the problem domain. For this reason, one needs a set of textbook problems. Each problem should cover a few components of the boundary value problem and a few possible features of the problem domain. This chapter contains such a set of textbook problems.

6.1 PROBLEMS IN ELECTROSTATICS

6.1.1 TWO COAXIAL CYLINDRICAL TUBES

In this section we will consider two infinitely long conducting coaxial cylindrical tubes depicted in Figure 6.1.1. The first (inner) tube is represented by its outer surface Γ_{D1}. The second (outer) tube is represented by its inner surface Γ_{D2}. The space between the tubes is empty. The first tube is at a potential of $\Phi = \Phi_0$ Volts. The second tube is at a potential of $\Phi = 0$ Volts, so the electro-

… Some pages have been omitted …

write:

$$\vec{K}_f = -\hat{r}\times\vec{M} = -\chi_m\hat{r}\times\vec{H} = -\frac{\chi_m}{\mu}\hat{r}\times\vec{B} = -\frac{\chi_m}{\mu}\left(\frac{1}{2}\mu a J_f\right)\hat{r}\times\hat{\phi} = -\frac{\chi_m a}{2}J_f\hat{z}. \tag{6.2.5}$$

The volume bound-current density is given by equation (1.3.15). Then by observing equations (1.3.19), (1.3.15), and (ii) in (1.3.22) we write:

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = \chi_m\vec{\nabla}\times\vec{H} = \chi_m\vec{J}_f = \chi_m J_f\hat{z}. \tag{6.2.6}$$

By substituting equations (6.2.5) and (6.2.6) into (6.2.4) we deduce that

$$I_b = 2\pi a\left(\left(-\frac{\chi_m a}{2}J_f\hat{z}\right)\cdot\hat{z}\right) + \pi a^2\left(\left(\chi_m J_f\hat{z}\right)\cdot\hat{z}\right) = -\pi a^2\chi_m J_f + \pi a^2\chi_m J_f = 0.$$

Therefore, the net total bound current encompassed by the Amperian loop nr. 2 in Figure 6.2.1 equals zero. Consequently, the magnetic field outside the wire is indeed independent of the magnetization of the wire as in this particular configuration the magnetic field induced by volume bound currents cancels out the magnetic field induced by the surface bound currents.

6.2.2 Spherical shield in a uniform magnetic field

In this section we will consider a magnetic shield shaped as a spherical shell. The shield is shown in Figure 6.2.2. It is made of soft permeable material of permeability μ. The permeability of the space inside and outside the shield is μ_0. The shield is exposed to a uniform magnetic field aligned with the z-axis. That is, the magnetic field is described by the following equation:

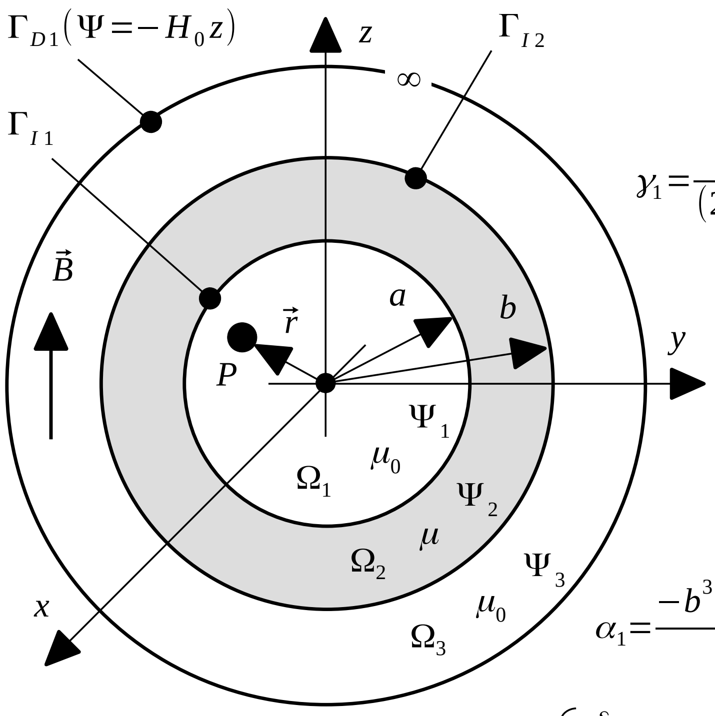
$$\vec{B} = B_0\hat{k} \tag{6.2.7}$$

in absence of the shield. We would like to obtain closed-form expressions for the total magnetic scalar potential Ψ, auxiliary vector field $\vec{H}$, and the magnetic field $\vec{B}$.

The sources of the magnetic field fall outside the problem domain. Therefore, we can apply the total magnetic scalar potential described in section 3.1. We adapt the boundary-value problem (3.1.14) as the following:

$$\begin{array}{rlll} \nabla^2\Psi = 0 & \text{in} & \Omega & \text{(i)}, \\ \Psi_3 = -H_0 z & \text{on} & \Gamma_{D1} & \text{(ii)}, \\ \Psi_1 = \Psi_2 & \text{on} & \Gamma_{I1} & \text{(iii)}, \\ \mu_0\hat{n}\cdot\vec{\nabla}\Psi_1 = \mu\hat{n}\cdot\vec{\nabla}\Psi_2 & \text{on} & \Gamma_{I1} & \text{(iv)}, \\ \Psi_2 = \Psi_3 & \text{on} & \Gamma_{I2} & \text{(v)}, \\ \mu\hat{n}\cdot\vec{\nabla}\Psi_2 = \mu_0\hat{n}\cdot\vec{\nabla}\Psi_3 & \text{on} & \Gamma_{I2} & \text{(vi)}. \end{array} \tag{6.2.8}$$

This adaptation deserves a few words. The magnetic permeability in all three subdomains, i.e., in Ω_1, Ω_2, and Ω_3, is assumed to be homogeneous. As soon as we will treat the three subdomains separately, the magnetic permeability



$$\Omega=\frac{(\mu_r-1)a^3}{(\mu_r+2)b^3}$$

$$\gamma_1=\frac{-3b^3H_0\Omega}{(2\mu_r+1)-2(\mu_r-1)\Omega}$$

$$\beta_1=\frac{(2\mu_r+1)\gamma_1}{(\mu_r-1)a^3}$$

$$\delta_1=\frac{\mu_r a^3\beta_1-2\mu_r\gamma_1}{a^3}$$

$$\alpha_1=\frac{-b^3H_0+2\mu_r\gamma_1-\mu_r b^3\beta_1}{2}$$

$$\Psi=\begin{cases}\delta_1 z & \text{if} \quad r\le a\\ \beta_1 z+\gamma_1 z r^{-3} & \text{if} \quad a\le r\le b\\ -H_0 z+\alpha_1 z r^{-3} & \text{if} \quad r\ge b\end{cases}$$

$$\vec{\nabla}\Psi=\begin{cases}\delta_1\hat{k} & \text{if} \quad r\le a\\ -3\gamma_1 z r^{-5}(x\hat{i}+y\hat{j}+z\hat{k})+(\beta_1+\gamma_1 r^{-3})\hat{k} & \text{if} \quad a\le r\le b\\ -3\alpha_1 z r^{-5}(x\hat{i}+y\hat{j}+z\hat{k})+(-H_0+\alpha_1 r^{-3})\hat{k} & \text{if} \quad r\ge b\end{cases}$$

Figure 6.2.2: A magnetic shield shaped as a spherical shell. The shield is shown in grey color. The outermost sphere in the figure represents infinity. The shield is exposed to a uniform magnetic field $\vec{B}$ that points in $z-$ direction.

can be considered to be a real-valued constant in each subdomain. Consequently, we can replace equation (i) in (3.1.14) with the Laplace equation (i) in (6.2.8). Equations (iii)-(vi) in (6.2.8) are interface conditions on the interfaces Γ_{I1} and Γ_{I2}. The Dirichlet condition (ii) in (6.2.8) can be deduced as the following. The magnetic field induced by the magnetization of the magnetic material of the shield vanishes at infinity. Consequently, the total magnetic field at infinity equals the applied magnetic field, i.e., the uniform magnetic field, $\vec{B}=B_0\hat{k}$. Therefore, taking into account (3.1.1) and (1.3.21) we can deduce the total scalar magnetic potential at infinity:

$$\Psi_3=-\frac{1}{\mu_0}B_0 z=-H_0 z.$$

The problem domain is spherically symmetric. Therefore, we will look for a solution expressed in the spherical coordinate system. The spherical coordinate system shown in Figure A.0.1. The spherical symmetry of the problem suggests that the total magnetic scalar potential is independent of ϕ coordinate. Consequently, the ϕ-components of the relevant vector fields, i.e., $\vec{B}$ and $\vec{H}$ equal zero. The method of separation of variables yields the following general solution to the Laplace's equation in terms of spherical coordinates, see, for instance, page 143 in [22]:

$$\Psi(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n \frac{1}{r^{n+1}} \right) P^{(n)}\big(\cos(\theta)\big), \tag{6.2.9}$$

where $P^{(n)}(x)$ are Legendre polynomials.

Let us adapt the general solution above separately for the three subdomains. We begin with the outermost subdomain, Ω_3. We can express the total magnetic potential at infinity by taking a limit $r \to \infty$ of the last equation:

$$\Psi_3 = \sum_{n=0}^{\infty} A_n r^n P^{(n)}\big(\cos(\theta)\big) \quad \text{on} \quad \Gamma_{D_1}. \tag{6.2.10}$$

In order to satisfy the Dirichlet condition (ii) in (6.2.8), we need to assume that

$$A_n = \begin{cases} -H_0 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases} \tag{6.2.11}$$

Then observing that $P^{(1)}\big(\cos(\theta)\big) = \cos(\theta)$ and $z = r\cos(\theta)$ we can rewrite (6.2.10) as

$$\Psi_3 = -H_0 r \cos(\theta) = -H_0 z \quad \text{on} \quad \Gamma_{D_1}.$$

Therefore, (6.2.11) ensures that the Dirichlet boundary condition (ii) in (6.2.8) is satisfied. Then equation (6.2.9) in the subdomain Ω_3 becomes

$$\Psi_3 = -H_0 r \cos(\theta) + \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P^{(n)}\big(\cos(\theta)\big) \quad \text{in} \quad \Omega_3. \tag{6.2.12}$$

In order to avoid a confusion, we shall rename the coefficients B_n in the subdomain Ω_3 to α_n. Then the last equation can be rewritten as

$$\Psi_3 = -H_0 r \cos(\theta) + \sum_{n=0}^{\infty} \alpha_n \frac{1}{r^{n+1}} P^{(n)}\big(\cos(\theta)\big) \quad \text{in} \quad \Omega_3. \tag{6.2.13}$$

Next, let us consider the second subdomain, Ω_2. In this subdomain we rename the coefficients A_n and B_n to β_n and γ_n. Then equation (6.2.9) can be rewritten as

$$\Psi_2 = \sum_{n=0}^{\infty} \left(\beta_n r^n + \gamma_n \frac{1}{r^{n+1}} \right) P^{(n)}\big(\cos(\theta)\big) \quad \text{in} \quad \Omega_2. \tag{6.2.14}$$

Next, let us turn our attention to the innermost subdomain, Ω_1. In order to avoid a singularity at the origin we have to set all coefficients B_n in (6.2.9) to zero. We rename the coefficients A_n in the subdomain Ω_1 to δ_n. Then equation (6.2.9) can be rewritten as

$$\Psi_1 = \sum_{n=0}^{\infty} \delta_n r^n P^{(n)}\big(\cos(\theta)\big) \quad \text{in} \quad \Omega_1. \tag{6.2.15}$$

The interface condition (v) in (6.2.8) implies that equations (6.2.13) and (6.2.14) must be equal on the interface Γ_{I_2},

$$\Psi_2\Big|_{r=b} = \Psi_3\Big|_{r=b}. \tag{6.2.16}$$

On the other hand, interface condition (vi) in (6.2.8) implies that the derivatives of (6.2.13) and (6.2.14) in the radial direction on the interface Γ_{I_2} must be scaled versions of each other,

$$\mu_r \frac{\partial \Psi_2}{\partial r}\Big|_{r=b} = \frac{\partial \Psi_3}{\partial r}\Big|_{r=b}, \tag{6.2.17}$$

where

$$\mu_r = \frac{\mu}{\mu_0}.$$

Similarly, the interface condition (iii) in (6.2.8) implies that equations (6.2.14) and (6.2.15) must be equal on the interface Γ_{I_1},

$$\Psi_1\Big|_{r=a} = \Psi_2\Big|_{r=a}. \tag{6.2.18}$$

On the other hand, interface condition (iv) in (6.2.8) implies that the derivatives of (6.2.14) and (6.2.15) in the radial direction on the interface Γ_{I_1} must be scaled versions of each other,

$$\frac{\partial \Psi_1}{\partial r}\Big|_{r=a} = \mu_r \frac{\partial \Psi_2}{\partial r}\Big|_{r=a}. \tag{6.2.19}$$

The interface conditions (6.2.16), (6.2.17), (6.2.18), and (6.2.19) can be satisfied by setting the coefficients α_n, β_n, γ_n, δ_n to zero for all $n \neq 1$ and then calculating proper values for α_1, β_1, γ_1, and δ_1. By setting all coefficients to zero for $n \neq 1$ we can rewrite equations (6.2.13) (6.2.14) and (6.2.15) as

$$\Psi_3 = \left(-H_0 r + \alpha_1 \frac{1}{r^2} \right) \cos(\theta) \quad \text{in} \quad \Omega_3, \tag{6.2.20}$$

$$\Psi_2 = \left(\beta_1 r + \gamma_1 \frac{1}{r^2} \right) \cos(\theta) \quad \text{in} \quad \Omega_2, \tag{6.2.21}$$

and

$$\Psi_1 = \delta_1 r \cos(\theta) \quad \text{in} \quad \Omega_1. \tag{6.2.22}$$

Next, let us calculate the four coefficients, α_1, β_1, γ_1, and δ_1. A straightforward application of interface condition (6.2.16) to equations (6.2.20) and (6.2.21) yields the following linear equation:

$$\beta_1 b + \gamma_1 b^{-2} = -H_0 b + \alpha_1 b^{-2}.$$

After rearranging the terms the last equation becomes

$$\alpha_1 - b^3 \beta_1 - \gamma_1 = b^3 H_0. \tag{6.2.23}$$

Similarly, a straightforward application of interface condition (6.2.17) to equations (6.2.20) and (6.2.21) yields the following linear equation:

$$\mu_r \beta_1 - 2\mu_r \gamma_1 b^{-3} = -H_0 - 2\alpha_1 b^{-3}.$$

After rearranging the terms the last equation becomes

$$2\alpha_1 + \mu_r b^3 \beta_1 - 2\mu_r \gamma_1 = -b^3 H_0. \tag{6.2.24}$$

A straightforward application of interface condition (6.2.18) to equations (6.2.21) and (6.2.22) yields the following linear equation:

$$\beta_1 a + \gamma_1 a^{-2} = \delta_1 a.$$

After rearranging the terms the last equation becomes

$$a^3 \beta_1 + \gamma_1 - a^3 \delta_1 = 0. \tag{6.2.25}$$

Similarly, application of interface condition (6.2.19) to equations (6.2.21) and (6.2.22) yields the following linear equation:

$$\mu_r \beta_1 - 2\mu_r \gamma_1 a^{-3} = \delta_1.$$

After rearranging the terms the last equation becomes

$$\mu_r a^3 \beta_1 - 2\mu_r \gamma_1 - a^3 \delta_1 = 0. \tag{6.2.26}$$

The four linear equations (6.2.23), (6.2.24), (6.2.25), and (6.2.26) can be written in a matrix form as

$$\begin{bmatrix} 2 & \mu_r b^3 & -2\mu_r & 0 \\ 1 & -b^3 & -1 & 0 \\ 0 & a^3 & 1 & -a^3 \\ 0 & \mu_r a^3 & -2\mu_r & -a^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} -b^3 H_0 \\ b^3 H_0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.2.27}$$

Next, we solve this system of linear equations by Gaussian elimination. The steps of the Gaussian elimination are shown in Figure 6.2.3. The results of the

	α_1	β_1	γ_1	δ_1	rhs
1	2	$\mu_r b^3$	$-2\mu_r$	0	$-b^3H_0$
2	1	$-b^3$	-1	0	b^3H_0
3	0	a^3	1	$-a^3$	0
4	0	$\mu_r a^3$	$-2\mu_r$	$-a^3$	0

$$\begin{pmatrix} 2 & \mu_r b^3 & -2\mu_r & 0 \\ 1 & -b^3 & -1 & 0 \\ 0 & a^3 & 1 & -a^3 \\ 0 & \mu_r a^3 & -2\mu_r & -a^3 \end{pmatrix} \qquad \begin{pmatrix} -b^3H_0 \\ b^3H_0 \\ 0 \\ 0 \end{pmatrix}$$

	α_1	β_1	γ_1	δ_1	rhs
1	2	$\mu_r b^3$	$-2\mu_r$	0	b^3H_0
2	1	$-b^3$	-1	0	$-b^3H_0$
3	0	$(\mu_r-1)a^3$	$-2\mu_r-1$	0	0
4	0	$\mu_r a^3$	$-2\mu_r$	$-a^3$	0

	γ_1	β_1	α_1	δ_1	rhs
1	$-2\mu_r$	$\mu_r b^3$	2	0	$-b^3H_0$
2	-1	$-b^3$	1	0	b^3H_0
3	$-2\mu_r-1$	$(\mu_r-1)a^3$	0	0	0
4	$-2\mu_r$	$\mu_r a^3$	0	$-a^3$	0

	γ_1	β_1	α_1	δ_1	rhs
1	$-2\mu_r$	$\mu_r b^3$	2	0	$-b^3H_0$
2	$-2\mu_r+2$	$(\mu_r+2)b^3$	0	0	$-3b^3H_0$
3	$-2\mu_r-1$	$(\mu_r-1)a^3$	0	0	0
4	$-2\mu_r$	$\mu_r a^3$	0	$-a^3$	0

	γ_1	β_1	α_1	δ_1	rhs
2	$-2\mu_r+2$	$(\mu_r+2)b^3$	0	0	$-3b^3H_0$
3	$-2\mu_r-1$	$(\mu_r-1)a^3$	0	0	0
1	$-2\mu_r$	$\mu_r b^3$	2	0	$-b^3H_0$
4	$-2\mu_r$	$\mu_r a^3$	0	$-a^3$	0

$$\Omega = \frac{(\mu_r-1)}{(\mu_r+2)}\frac{a^3}{b^3}$$

$\Omega = (\mu_r-1)\,a^3/(\mu_r+2)b^3$

	γ_1	β_1	α_1	δ_1	rhs
2	$(-2\mu_r+2)\Omega-(-2\mu_r-1)$	0	0	0	$-3b^3H_0\Omega$
3	$-2\mu_r-1$	$(\mu_r-1)a^3$	0	0	0
1	$-2\mu_r$	$\mu_r b^3$	2	0	$-b^3H_0$
4	$-2\mu_r$	$\mu_r a^3$	0	$-a^3$	0

$$\gamma_1 = \frac{-3b^3H_0\Omega}{(2\mu_r+1)-2(\mu_r-1)\Omega}$$

$$\beta_1 = \frac{(2\mu_r+1)\gamma_1}{(\mu_r-1)a^3}$$

$$\delta_1 = \frac{\mu_r a^3\beta_1 - 2\mu_r\gamma_1}{a^3} \qquad \alpha_1 = \frac{-b^3H_0 + 2\mu_r\gamma_1 - \mu_r b^3\beta_1}{2}$$

Figure 6.2.3: Solution to the system of linear equations (6.2.27) by Gaussian elimination.

elimination process are:

$$\Omega = \frac{(\mu_r - 1)}{(\mu_r + 2)} \frac{a^3}{b^3},$$
$$\gamma_1 = \frac{-3b^3 H_0 \Omega}{(2\mu_r + 1) - 2(\mu_r - 1)\Omega},$$
$$\beta_1 = \frac{(2\mu_r + 1)\gamma_1}{(\mu_r - 1)a^3},$$
$$\alpha_1 = \frac{-b^3 H_0 + 2\mu_r \gamma_1 - \mu_r b^3 \beta_1}{2},$$
$$\delta_1 = \frac{\mu_r a^3 \beta_1 - 2\mu_r \gamma_1}{a^3}.$$

Finally, we can summarize equations (6.2.20), (6.2.21), and (6.2.22) as

$$\Psi = \begin{cases} \delta_1 r \cos(\theta) & \text{if} \quad r \leq a \\ \left(\beta_1 r + \dfrac{\gamma_1}{r^2}\right)\cos(\theta) & \text{if} \quad a \leq r \leq b \\ \left(-H_0 r + \dfrac{\alpha_1}{r^2}\right)\cos(\theta) & \text{if} \quad r \geq b. \end{cases} \tag{6.2.28}$$

It is convenient to convert equation (6.2.28) in the Cartesian coordinate system. From Figure A.0.1 we deduce that $z = r\cos(\theta)$. Then (6.2.28) can be rewritten in the Cartesian coordinate system as

$$\Psi = \begin{cases} \delta_1 z & \text{if} \quad r \leq a \\ \beta_1 z + \gamma_1 \dfrac{z}{r^3} & \text{if} \quad a \leq r \leq b \\ -H_0 z + \alpha_1 \dfrac{z}{r^3} & \text{if} \quad r \geq b, \end{cases} \tag{6.2.29}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}.$$

The gradient of the total magnetic scalar potential in Cartesian coordinate system can be obtained by a straightforward differentiation of (6.2.29):

$$\vec{\nabla}\Psi = \begin{cases} \delta_1 \hat{k} & \text{if} \quad r \leq a \\ -3\gamma_1 \dfrac{z}{r^5}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) + \left(\beta_1 + \gamma_1 \dfrac{1}{r^3}\right)\hat{k} & \text{if} \quad a \leq r \leq b \\ -3\alpha_1 \dfrac{z}{r^5}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) + \left(-H_0 + \alpha_1 \dfrac{1}{r^3}\right)\hat{k} & \text{if} \quad r \geq b. \end{cases} \tag{6.2.30}$$

The auxiliary vector field $\vec{H}$ and the magnetic field $\vec{B}$ can be calculated as

$$\vec{H} = -\vec{\nabla}\Psi$$

and

$$\vec{B} = -\mu\vec{\nabla}\Psi.$$

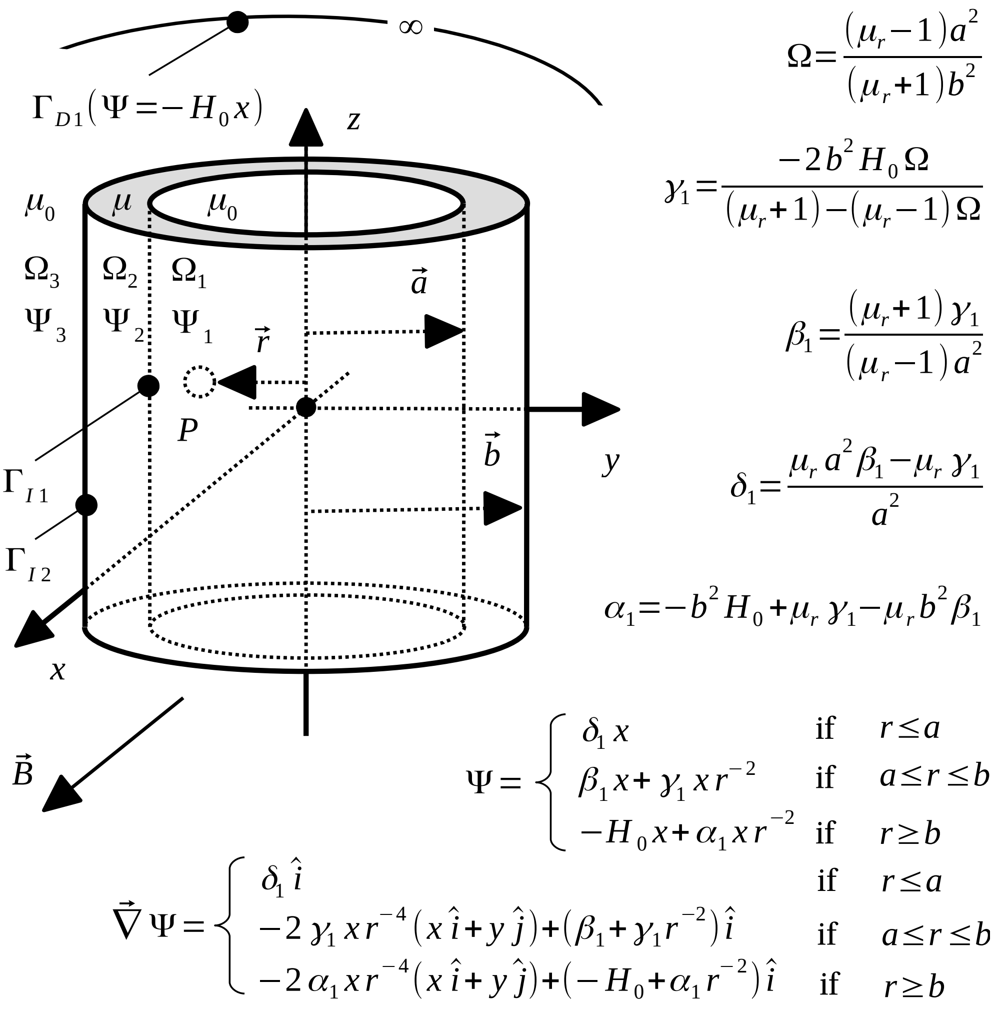


Figure 6.2.4: A cylindrical shield made of soft magnetic material in a free unbounded space. The cross section of the shield is shown in grey color. The shield is exposed to a uniform magnetic field $\vec{B}$ that points in x direction. The arc in the top part of the figure, Γ_{D1}, represents infinity.

6.2.3 Cylindrical shield in a uniform magnetic field

In this section we will consider a magnetic shield shaped as an infinitely long cylindrical tube coaxial with the z axis. The shield is shown in Figure 6.2.4. It is made of soft permeable material of permeability μ. The permeability of the space inside and outside the shield is μ_0. The shield is exposed to a uniform magnetic field aligned with the x axis. That is, the magnetic field is described by the following equation in absence of the shield:

$$\vec{B} = B_0 \hat{i}.$$

We would like to obtain closed-form expressions for: the total magnetic potential Ψ, auxiliary vector field $\vec{H}$, and magnetic field $\vec{B}$.

This problem is similar to the problem described in the preceding section. We adapt the boundary value problem (6.2.8) by changing only the Dirichlet

… Some pages have been omitted …

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References

[1] Adams, R. A., and Essex, C. *Calculus: A Complete Course,.* Pearson Education, 2013.

[2] Ball, E. Potential from a ring of charge. In *Proceedings of the Institution of Electrical Engineers* (1977), vol. 124, IET, p. 664.

[3] Bangerth, W., Hartmann, R., and Kanschat, G. deal.II-a general-purpose object-oriented finite element library. *ACM Transactions on Mathematical Software (TOMS) 33*, 4 (2007), 24–es.

[4] Bayliss, A., Gunzburger, M., and Turkel, E. Boundary conditions for the numerical solution of elliptic equations in exterior regions. *Siam Journal on Applied Mathematics 42* (1982), 430–451.

[5] Bedrosian, G. Magnetostatic cancellation error revisited. *IEEE transactions on magnetics 27*, 5 (1991), 4181–4184.

[6] Bermant, A. *Mappings. Curvilinear coordinates. Transformations. Green's Formulas.* GIFML, Moscow, 1958.

[7] Biro, O., Preis, K., and Richter, K. R. On the use of the magnetic vector potential in the nodal and edge finite element analysis of 3d magnetostatic problems. *IEEE Transactions on magnetics 32*, 3 (1996), 651–654.

[8] Bossavit, A. A rationale for 'edge-elements' in 3-d fields computations. *IEEE Transactions on Magnetics 24* (1988), 74–79.

[9] Bossavit, A. *Computational electromagnetism: variational formulations, complementarity, edge elements.* Academic Press, 1998.

[10] Bossavit, A. On the geometry of electromagnetism (4): Maxwell's house. *Journal of the Japan Society of Applied Electromagnetics 6*, 4 (1998), 318–326.

[11] Bueler, E. *PETSc For partial differential equations: Numerical solutions in C and Python.* SIAM, 2020.

[12] CHEN, Q., AND KONRAD, A. A review of finite element open boundary techniques for static and quasi-static electromagnetic field problems. *IEEE Transactions on Magnetics 33* (1997), 663–676.

[13] CIARLET, P. G. *The finite element method for elliptic problems.* North-Holland Publishing Company, 1978.

[14] COEY, J. M. D. *Magnetism and magnetic materials.* Cambridge university press, 2010.

[15] DEBNATH, L., AND MIKUSINSKI, P. *Introduction to Hilbert spaces with applications.* Elsevier Academic press, 2005.

[16] DEMKOWICZ, L. F., KURTZ, J., PARDO, D., PASZYŃSKI, M., RACHOWICZ, W., AND ZDUNEK, A. *Computing with hp-ADAPTIVE FINITE ELEMENTS: Volume 1 One and Two Dimensional Elliptic and Maxwell Problems.* 2006.

[17] ELSGOLC, L. D. *Calculus of Variations.* Dover Books on Mathematics. Dover Publications, 2007.

[18] FEYNMAN, R. P., LEIGHTON, R. B., AND SANDS, M. *The Feynman Lectures on Physics*, vol. 1. Addison-Wesley, Reading, 1977.

[19] FLEISCH, D. A. *A student's guide to vectors and tensors.* Cambridge University Press, 2011.

[20] GOCKENBACH, M. S. *Understanding and implementing the finite element method*, vol. 97. Siam, 2006.

[21] GRATKOWSKI, S., STAWICKI, K., AND ZIÓLKOWSKI, M. General closed-form asymptotic boundary conditions for finite element analysis of exterior electrical field problems. *Przegląd Elektrotechniczny 1* (2016), 15–18.

[22] GRIFFITHS, D. J. *Introduction to electrodynamics*, 4 ed. Cambridge University Press, 2017.

[23] HEAVISIDE, O. *Electromagnetic theory*, vol. 1. The Electrician Printing and Publishing Co, London, 1893.

[24] JACKSON, J. D. *Classical electrodynamics*, 3 ed. John Wiley & Sons, Inc., 1998.

[25] JIN, J.-M. *The finite element method in electromagnetics*, 3 ed. Wiley-IEEE Press, 2014.

[26] KONRAD, A., AND GRAOVAC, M. The finite element modeling of conductors and floating potentials. *IEEE Transactions on Magnetics 32* (1996), 4329–4331.

[27] KUCZMANN, M. *Potential formulations in magnetics applying the finite element method.* Lecture notes, Laboratory of Electromagnetic Fields,"Széchenyi István" University, Gyor, Hungary, 2009.

[28] LOGG, A., MARDAL, K.-A., AND WELLS, G. *Automated solution of differential equations by the finite element method: The FEniCS book*, vol. 84. Springer Science & Business Media, 2012.

[29] MEDVINSKY, M., TSYNKOV, S., AND TURKEL, E. Direct implementation of high order bgt artificial boundary conditions. *J. Comput. Phys. 376* (2019), 98–128.

[30] MIKHLIN, S. G. *Variational Methods in Mathematical Physics.* Nauka, Moscow, 1970.

[31] MIKHLIN, S. G., AND SMOLITSKI, H. L. *Approximate methods for solution of differential and integral equations.* Nauka, Moscow, 1965.

[32] PHILLIPS, G. M. *Interpolation and approximation by polynomials.* Springer-Verlag, 2003.

[33] REDDY, J. N. *Energy principles and variational methods in applied mechanics*, 2 ed. John Wiley & Sons, 2002.

[34] REN, Z. Influence of the rhs on the convergence behaviour of the curl-curl equation. *IEEE Transactions on Magnetics 32* (1996), 655–658.

[35] RINCON, D., AGUILERA, E., AND CHACÓN, J. C. Numerical treatment of floating conductors based on the traditional finite element formulation. *Advanced Electromagnetics 7*, 3 (2018), 46–55.

[36] SADIKU, M. N. O., AND NELATURY, S. R. *Analytical Techniques in Electromagnetics.* CRC Press, 2015.

[37] SCHEY, H. M. *Div, grad, curl, and all that: an informal text on vector calculus.* WW Norton New York, 2005.

[38] SCHUTZ, B. *A first course in general relativity*, 2 ed. Cambridge university press, 2009.

[39] SILVESTER, P. P., AND DUFRESNE, M. Computational error in reduced scalar magnetic potential problems. *COMPEL-The international journal for computation and mathematics in electrical and electronic engineering* (1992).

[40] SIMKIN, J., AND TROWBRIDGE, C. On the use of the total scalar potential on the numerical solution of fields problems in electromagnetics. *International journal for numerical methods in engineering 14*, 3 (1979), 423–440.

[41] STRANG, G. *Introduction to linear algebra*, 5 ed. Wellesley-Cambridge Press, 2016.

[42] STRATTON, J. A. *Electromagnetic theory*. McGraw-hill book company, 1941.

[43] VETTERLING, W. T., PRESS, W. H., TEUKOLSKY, S. A., AND FLANNERY, B. P. *Numerical recipes. The Art of Scientific Computing*, 3 ed. Cambridge University Press, 2007.

[44] ZIENKIEWICZ, O. C., TAYLOR, R. L., AND ZHU, J. Z. *The finite element method: its basis and fundamentals*. Elsevier, 2005.